Quiz 2 and Exam 1 review (draft: 2019/02/17-15:51:18)

Quiz 2 covers most of chapter 2, all probability laws/rules up to and including independence, partitions, conditional probability, and Baye's theorem.

Exam 1 covers all material since the first day of class.

Example/practice problems:

1. A batch of fuses produced by a particular factory are tested. There are two types of fuses. 55% of the fuses are type I, and 45% are type II. 5% of the type I fuses fail to meet quality specifications, and 4% of type II fail.

   (a) What is the probability a type I fuse satisfies quality specifications?
   Solution:
   Let T_i = the event that a fuse is type i. We use the fact that fuse type is a partition: $S = T_1 \cup T_2$. Let A = the event that a fuse satisfies quality specs. We want to calculate $P(A \mid T_1)$. We are given $P(A^c \mid T_1) = 0.05$ thus by the probability rule for complements $P(A \mid T_1) = 1 - P(A^c \mid T_1) = 0.95$.

   (b) If a fuse is randomly selected, what is the probability it is of type I and satisfied quality specs?
   Solution:
   $P(A \cap T_1) = P(A \mid T_1)P(T_1) = (0.95)(0.55) = 0.5225$

   (c) What is the probability a randomly selected fuse satisfies quality specs?
   Solution:

$$\begin{aligned} P(A) &= P(A \cap T_1) + P(A \cap T_2) \\ &= P(A \mid T_1)P(T_1) + P(A \mid T_2)P(T_2) \\ &= (0.95)(0.55) + (0.96)(0.45) \\ &= 0.9545. \end{aligned}$$

   Or in more plain language:

$$\begin{aligned} P(\text{satisfy}) &= P(\text{satisfy} \cap \text{type I}) + P(\text{satisfy} \cap \text{type II}) \\ &= P(\text{satisfy} \mid \text{type I})P(\text{type I}) + P(\text{satisfy} \mid \text{type II})P(\text{type II}) \end{aligned}$$

   (d) Given that we know a particular fuse has failed to meet quality specs, what is the probability it is type I?
   Solution:
   Now this will involve Baye's theorem since we know $P(A^c \mid T_1)$ but want to calculate $P(T_1 \mid A^c)$.

$$\begin{aligned} P(T_1 \mid A^c) &= \frac{P(A^c \cap T_1)}{P(A^c)} \\ &= \frac{P(A^c \mid T_1)P(T_1)}{P(A^c \mid T_1)P(T_1) + P(A^c \mid T_2)P(T_2)} \\ &= \frac{(0.05)(0.55)}{(0.05)(0.55) + (0.04)(0.45)} \\ &= \frac{275}{275 + 180} \\ &\approx 0.6044 \end{aligned}$$

(e) Are fuse type and whether it passes quality specs independent?

Solution:

$P(A) = 0.9545$ and $P(T_1) = 0.55$ but $P(A \cap T_1) = 0.5225$. This is not the same as $P(A) \cdot P(T_1) = (0.9545) \cdot (0.55) = 0.524975$. Thus they are not independent.

There are many ways to check this:
$P(A) = 0.9545 \neq 0.95 = P(A \mid T_1)$.
$P(A) = 0.9545 \neq 0.96 = P(A \mid T_2)$.

2. We roll a pair of 6-sided dice. Let $A = \{\text{sum is even}\}$, and $B = \{\text{both dice are even}\}$. Calculate $P(A)$, $P(B)$, $P(A \mid B)$, and $P(B \mid A)$. Are A and B independent?

   Solution:

   Let's first list out all outcomes in these events. To save on space, I will list the outcomes as two-digit numbers, with the first digit being the first die and the second digit being the second die, e.g. 63 is when the first die is 6 and the second die is 3.

$$A = \{11, 33, 55, 13, 31, 15, 51, 35, 53, 22, 44, 66, 24, 42, 26, 62, 46, 64\}$$

$$B = \{22, 44, 66, 24, 42, 26, 62, 46, 64\}$$

   $P(A) = 18/36 = 0.5$, $P(B) = 9/36 = 0.25$, $P(A \mid B) = 1$ since $B \subset A$, and $P(B \mid A) = \frac{9}{18} = 0.5 \neq P(B)$ thus they are not independent.

3. Now let's roll a single 6-sided die. Let events: $A = \{2, 4, 6\} = \{\text{even}\}$, and $B = \{1, 2\}$, $C = \{1, 2, 3\}$, $D = \{1, 2, 3, 4\}$. Is A independent of any of the other 3 events?

   Solution:

   $P(A) = \frac{3}{6} = 0.5$.

   $P(A \mid B) = \frac{1}{2} = 0.5 = P(A)$.

   $P(A \mid C) = \frac{1}{3}$.

   $P(A \mid D) = \frac{2}{4} = 0.5 = P(A)$.

   Thus A and D are independent, and A and B are independent. A and C are not independent. Note that independence is a property that applies to a pair of events.

4. A particular political candidate is running for office. It is known that 20% of party D voters support the candidate, that 60% of party R voters support the candidate, and that 40% of independent (no party affiliation) voters support the candidate. The general population of voters is made up of equal proportions of each party but 20% of the population are independents. Given that a randomly selected voter supports the candidate, what is the probability they are a member of party D? Are political party membership and support for the candidate independent events?

   Solution:

   Let events $D = \{\text{voter is member of party D}\}$, $R = \{\text{voter is member of party R}\}$, $I = \{\text{voter is independent}\}$, and $S = \{\text{voter supports the candidate}\}$. We are given that $P(S \mid D) = 0.2$, $P(S \mid R) = 0.6$, $P(D) = P(R) = 0.40$, and $P(I) = 0.2$.

$$P(D \mid S) = \frac{P(S \mid D)P(D)}{P(S \mid D)P(D) + P(S \mid R)P(R) + P(S \mid I)P(I)}$$
$$= \frac{0.2 \cdot 0.4}{0.2 \cdot 0.4 + 0.6 \cdot 0.4 + 0.4 \cdot 0.2} = \frac{8}{8 + 24 + 8} = \frac{1}{5} = 20\%$$

$$P(R \mid S) = \frac{P(S \mid R)P(R)}{P(S \mid D)P(D) + P(S \mid R)P(R) + P(S \mid I)P(I)}$$
$$= \frac{0.6 \cdot 0.4}{0.2 \cdot 0.4 + 0.6 \cdot 0.4 + 0.4 \cdot 0.2} = \frac{24}{8 + 24 + 8} = \frac{3}{5} = 60\%$$

$$P(I \mid S) = \frac{P(S \mid I)P(I)}{P(S \mid D)P(D) + P(S \mid R)P(R) + P(S \mid I)P(I)}$$
$$= \frac{0.4 \cdot 0.2}{0.2 \cdot 0.4 + 0.6 \cdot 0.4 + 0.4 \cdot 0.2} = \frac{8}{8 + 24 + 8} = \frac{1}{5} = 20\%$$

Note that $P(D) = 40\% \neq P(D \mid S) = 20\%$, $P(R) = 40\% \neq P(R \mid S) = 60\%$, but $P(I) = 20\% = P(I \mid S)$. Thus, at least for this example, independent voters truly are *independent* in the probability theory sense!

Note that this is just a quirk of the numbers used in this problem. If you alter them slightly, the independence of independent voters goes away.

5. If you flip three fair coins, and there is at least one head, what is the probability that there are at least two heads?

   Solution:

   The only outcome excluded from {at least one head} is the all tails outcome, thus a total of $2^3 - 1 = 7$ outcomes. How many ways are there to get at least two heads? There are $\binom{3}{2}$ ways to get two heads and $\binom{3}{3}$ ways to get three heads, thus a total of 4 outcomes. All 4 of these outcomes also have at least one head. Thus the probability $P(\text{at least 2 H's} \mid \text{at least 1 H}) = \frac{4}{7}$.

6. Let's look at flipping two fair coins. Show that the coins are *independent*.

   Solution:

   That the coins are independent satisfies intuition. $S = \{HH, HT, TH, TT\}$ and we assume the outcomes are all equally likely. Let $A =$ {first coin is H}={HH,HT}, and $B =$ {second coin is H}={HH,TH}. So $P(A) = P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{4} = P(A) \cdot P(B)$.

   This is enough to show they are independent. It is a rule that when A and B are independent, then so are the complementary events A^c and B, A and B^c, and A^c and B^c.