HOMOLOGICAL CHARACTERIZATIONS OF QUASI-COMPLETE INTERSECTIONS

JASON M. LUTZ

Abstract. Let R be a commutative ring, $(\boldsymbol{f})$ an ideal of R, and $E = K(\boldsymbol{f}; R)$ the Koszul complex. We investigate the structure of the Tate construction T associated with E. In particular, we study the relationship between the homology of T, the quasi-complete intersection property of ideals, and the complete intersection property of (local) rings.

Introduction

Let $(R, \mathfrak{m})$ be a commutative, Noetherian, local ring with maximal ideal $\mathfrak{m}$ and let I be a proper, non-zero ideal of R. Fix a generating set $\boldsymbol{f}$ of I, and let E be the Koszul complex on $\boldsymbol{f}$.

Recall that I is a *complete intersection* ideal if it can be generated by a regular sequence. As R is local, this condition is tantamount to the following (equivalent) conditions:

(1) $H_i(E) = 0$ for all $i > 0$.
(2) $H_1(E) = 0$.

Let S denote R/I. There is a canonical homomorphism of graded S-algebras:

$$\begin{aligned} \lambda_*^S : \wedge_*^S H_1(E) &\to H_*(E), \\ \mathrm{cls}(z_1) \wedge \cdots \wedge \mathrm{cls}(z_m) &\mapsto \mathrm{cls}(z_1 \wedge \cdots \wedge z_m). \end{aligned}$$

The focus of this paper is on quasi-complete intersection ideals.

Definition. The ideal I is a said to be a *quasi-complete intersection* if $H_1(E)$ is free as an S-module and λ_*^S is an isomorphism.

As Avramov, Henriques, and Şega [5] note, these ideals were first introduced in Rodicio's paper [18] and in his joint work with Blanco and Majadas [9] as ideals having *free exterior Koszul homology*. The quasi-complete intersection nomenclature is due to Avramov et al. [5, 1.1].

Like complete intersection ideas, quasi-complete intersections can be described from an ideal-theoretic perspective: An ideal generated by a sequence of *exact elements* is necessarily a quasi-complete intersection; see [5, Theorem 3.7] and [15, Theorem 1.8]. The converse does not hold: In [16, Example 4.1], Kustin, Şega, and Vraciu give an example of a quasi-complete intersection which cannot be generated by a sequence of exact elements.

We study homological characterizations of quasi-complete intersections. Our primary tool in this study is the *Tate construction*. This complex is the second step in a *Tate resolution* of S over R, i.e, it is the result of adjoining (to the Koszul complex E) variables of degree two to annihilate the degree one homology of E; see [20, §2]. In Section 1 we recall the properties of the Tate construction and the related *Cartan construction*.

Blanco, Majadas and Rodicio [10, Theorem 1] provide a characterization of quasi-complete intersection ideals as follows. Let $\boldsymbol{z}$ denote a set set of cycles whose homology classes generate $H_1(E)$ and let T be the Tate construction on $\boldsymbol{f}$ and $\boldsymbol{z}$. Then $H_i(T) = 0$ for all $i > 0$ if and only if I is

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a quasi-complete intersection and $\boldsymbol{z}$ represents a basis for the S-module $H_1(E)$. In Section 2 we strengthen one direction of this characterization with the following result (Theorem 2.1):

Theorem A. *Suppose $\boldsymbol{f} = \{f_1, \dots, f_n\}$ and set $b = n - \operatorname{depth}(I, R)$. If $H_i(T) = 0$ for $i = 2, \dots, b+2$, then I is a quasi-complete intersection and $\boldsymbol{z}$ represents a basis for the S-module $H_1(E)$.*

In Section 3 we provide another characterization of quasi-complete intersection ideals (Theorem 3.1). We detect the quasi-complete intersection property from a band of vanishing of width $b+1$ (as in Theorem A) and with an additional assumption on the size of $\boldsymbol{z}$ we have flexibility in the location of the band.

In Section 4 we utilize the fact (established by Assmus [1, Theorem 2.7]) that the maximal ideal of a local ring is a quasi-complete intersection if and only if the ring is a complete intersection. We obtain the following characterization of (local) complete intersection rings (Theorem 4.4):

Theorem B. *Let $\boldsymbol{x}$ be a minimal generating set of $\mathfrak{m}$, and set $b = \operatorname{embdim} R - \operatorname{depth} R$. Suppose $\boldsymbol{z} = \{z_1, \dots, z_b\}$ is a set of cycles whose homology classes form a minimal generating set of $H_1(E)$. Let T be the Tate construction on $\boldsymbol{x}$ and $\boldsymbol{z}$. The following conditions are equivalent:*

(1) R is a complete intersection.
(2) There exists an integer $q \geq 2$ such that $H_i(T) = 0$ for $i \in \{q, \dots, q+b-1\}$.

In particular, the quasi-complete intersection property of $\mathfrak{m}$ (equivalently: the complete intersection property of R) can be detected from a band of vanishing of $H_*(T)$ of width b (compared to width $b+1$ of Theorems A and B).

Assmus [1, Theorem 2.7] also characterizes complete intersections as rings for which $H_2(T) = 0$. Utilizing results of Halperin [13, Theorem B], Gulliksen [12, Theorem 3.5.1], and Avramov [3, Theorem 2.3], we expand on this characterization with the following result (Theorems 4.7 and 5.5):

Theorem C. *Suppose that one of the following conditions holds:*

(1) $H_i(T) = 0$ for $i = 3$ or 4.
(2) $H_i(T) = 0$ for some $i \geq 5$ and there is a Golod homomorphism from a complete intersection ring onto $\widehat{R}$.

Then R is a complete intersection.

1. The Tate Construction

Throughout this paper, R is a commutative (not necessarily Noetherian) ring. We recall the construction of two families of complexes, due respectively to Tate [20] and Cartan. We adopt the notation of [12, 20]. In particular, if X is a differential graded (DG) R-algebra and v is a homogeneous cycle of X, then $X\langle V \mid \partial V = v\rangle$ denotes the extension of X obtained by adjoining a variable V to annihilate the cycle v. The type of variable depends on the degree of v: If $|v|$ is odd then V is an exterior variable, and if $|v|$ is even then V is a *divided powers variable*; see [4, Construction 6.1].

Let I denote a proper non-zero ideal of R, and $S = R/I$. We fix a generating set $\boldsymbol{f}$ of I. Let E denote the Koszul complex on $\boldsymbol{f}$, i.e., $E = K(\boldsymbol{f}; R)$. We have an identification of E as an extension of R. Let $\boldsymbol{u} = \{u_f : f \in \boldsymbol{f}\}$ denote a set of degree one exterior variables. Then

$$E = R\langle \boldsymbol{u} \mid \partial u_f = f\rangle.$$

Construction 1.1. *The Tate construction.* Let $\boldsymbol{z}$ be a set of cycles of degree one such that the homology classes $\{\operatorname{cls}(z) : z \in \boldsymbol{z}\}$ generate $H_1(E)$. Let $\boldsymbol{w} = \{w_z : z \in \boldsymbol{z}\}$ denote a set of degree two divided powers variables. The *Tate construction on $\boldsymbol{f}$ and $\boldsymbol{z}$*, denoted $T(\boldsymbol{f}; \boldsymbol{z})$ is

$$\begin{aligned} T(\boldsymbol{f}; \boldsymbol{z}) &= R\langle \boldsymbol{u}, \boldsymbol{w} \mid \partial u_f = f, \, \partial w_z = z\rangle \\ &= E\langle \boldsymbol{w} \mid \partial w_z = z\rangle. \end{aligned}$$

Let T denote the Tate construction $T(\boldsymbol{f};\boldsymbol{z})$; we have the equality $H_1(T)=0$ and isomorphisms $H_0(T)\cong H_0(E)\cong S$.

The Tate construction T has the following explicit presentation. Let W be a graded R-module on the basis $\boldsymbol{w}$, and let Γ^R_*W denote the divided powers algebra on W. For integers $j(w)\geq 0$ with $p=\sum_{w\in\boldsymbol{w}} j(w)$, the distinct expressions $\prod_{w\in\boldsymbol{w}} w^{(j(w))}$ form a basis of Γ^R_pW. This yields the following presentation of the complex:

$$T_n=\bigoplus_{2p+q=n} E_q\otimes_R\Gamma^R_pW,$$

$$\partial^T_n\left(e\otimes\prod_{w\in\boldsymbol{w}} w^{(j(w))}\right)=\partial^E(e)\otimes\prod_{w\in\boldsymbol{w}} w^{(j(w))}$$
$$+(-1)^{|e|}\sum_{w'\in\boldsymbol{w}}\left(z'e\otimes w'^{(j(w')-1)}\prod_{w\neq w'} w^{(j(w))}\right).$$

Remark 1.2. For a local ring $(R,\mathfrak{m})$, we have a uniqueness property of the Tate construction. Let $\boldsymbol{f}$ denote a minimal generating set of I, and let E denote the Koszul complex on $\boldsymbol{f}$. If $\boldsymbol{z}$ is set of degree one cycles whose homology classes form a minimal generating set of $H_1(E)$, then $T(\boldsymbol{f};\boldsymbol{z})$ is unique up to isomorphism (see, for example [6, 1.2]). As such, we may (in this context) simply refer to the *Tate construction on I* without risk of confusion.

Remark 1.3. The explicit presentation of the Tate construction $T=T(\boldsymbol{f};\boldsymbol{z})$ yields a convergent first-quadrant spectral sequence:

$$\left\{d^{p,q}_r: E^r_{p,q}\to E^r_{p-r,q+r-1}\right\}_{r\geq 0};\qquad E^r_{pq}\implies H_{p+q}(T).$$

The E^0 and E^1 pages are as follows:

$$E^0_{pq}=E_{q-p}\otimes_R\Gamma^R_pW,\qquad E^1_{pq}=H_{q-p}(E)\otimes_S\Gamma^S_p(S\otimes_R W).$$

Let H_i denote $H_i(E)$ and let Γ_j denote $\Gamma^S_j(S\otimes_R W)$. For $q\geq 0$, the row E^1_{*q} of the E^1 page of the spectral sequence is as follows:

$$0\longleftarrow H_q\longleftarrow H_{q-1}\otimes_S\Gamma_1\longleftarrow\cdots\longleftarrow H_2\otimes_S\Gamma_{q-2}\longleftarrow H_1\otimes_S\Gamma_{q-1}\longleftarrow\Gamma_q\longleftarrow 0$$

This spectral sequence will be utilized in Section 2; the realization of the Tate construction as an extension of R will appear in Sections 3 through 5.

We now recall the prototype for the Tate construction: the Cartan construction.

Construction 1.4. *The Cartan construction.* Let B denote a DG R-algebra with differential $\partial^B=0$. Let $\boldsymbol{y}$ denote a set of generators of B_1, and let $\boldsymbol{w}=\{w_y : y\in\boldsymbol{y}\}$ denote a set of degree two divided powers variables. The *Cartan construction C on B* is the extension

$$C=B\langle\boldsymbol{w}\,|\,\partial w_y=y\rangle.$$

Remark 1.5. Let C be the Cartan construction on B. Then C is bigraded:

$$C_{p,q}=B_{q-p}\otimes_R\Gamma^R_pW,\qquad C_n=\bigoplus_{p+q=n} C_{p,q}.$$

Moreover C decomposes into *strands* $C_{*,q}$:

$$0\longleftarrow B_q\longleftarrow B_{q-1}\otimes_R\Gamma^R_1W\longleftarrow\cdots\longleftarrow B_1\otimes_R\Gamma^R_{q-1}W\longleftarrow\Gamma^R_qW\longleftarrow 0.$$

As such, we have a decomposition of the homology of C:

$$H_n(C) = \bigoplus_{p+q=n} H_p(C_{*,q}). \tag{1.6}$$

Remark 1.7. Let G be a free R-module on a basis $\boldsymbol{g}$. Set $B = \wedge_*^R G$, and consider B as a DG algebra with differential $\partial^B = 0$; note that $H(B) = B$. Let C be the Cartan construction on B. Then $H_p(C_{*,q}) = 0$ for all $(p, q) \neq (0, 0)$.

Indeed, $\boldsymbol{g}$ is *regular* on B (in the sense of [4, §6]) so that [4, Proposition 6.1.7] yields an isomorphism

$$\frac{B}{(\boldsymbol{g})B} \cong H(C).$$

In light of this and the equalities $(\boldsymbol{g})B = B_{\geq 1}$ and $B_0 = R$ we have $H_n(C) = 0$ for $n > 0$ and $H_0(C) = R$. Now (1.6) yields desired result.

2. Low-degree vanishing of $H_*(T)$

Recall that E is the Koszul complex on a fixed generating set $\boldsymbol{f}$ of I. Throughout this section, $\boldsymbol{z}$ denotes a set of degree one cycles such that the homology classes $\{\operatorname{cls}(z) : z \in \boldsymbol{z}\}$ generate $H_1(E)$. Let T denote the Tate construction on $\boldsymbol{f}$ and $\boldsymbol{z}$ given by Construction 1.1. In this section, we prove the following result (Theorem A):

Theorem 2.1. *Suppose that $I = (\boldsymbol{f})$ is a proper, non-zero ideal of R, and set $b = \max\{i : H_i(E) \neq 0\}$. Suppose $\boldsymbol{z}$ is a set of cycles whose homology classes generate $H_1(E)$. Let T be the Tate construction on $\boldsymbol{f}$ and $\boldsymbol{z}$. The following conditions are equivalent:*

(1) I is a quasi-complete intersection ideal and $\boldsymbol{z}$ represents a basis of the S-module $H_1(E)$.
(2) $H_i(T) = 0$ for all $i > 0$.
(3) $H_i(T) = 0$ for $i = 2, \dots, b + 2$.

Remark 2.2. When R is Noetherian, the integer b can be computed as follows: For $I = (f_1, \dots, f_c)$ and $I \neq I^2$, [17, Theorem 16.8] yields that $b = c - \operatorname{depth}(I, R)$, where $\operatorname{depth}(I, R)$ denotes the length of a maximal R-sequence contained in I.

We begin by noting a relationship between the properties of the map λ_*^S and the homology of T. The map $d_1^{1,1}$ of the spectral sequence of Remark 1.3 is given by

$$d_1^{1,1} : S \otimes_R W \to H_1(E), \quad s \otimes_R w_i \mapsto s \operatorname{cls}(z_i).$$

The construction of T yields that $d_1^{1,1}$ is surjective, $S \otimes_R W$ free over S, and $H_1(T) = 0$. In addition, $\lambda_1^S : \wedge_1^S H_1(E) \to H_1(E)$ is the identity map.

Proposition 2.3. *Let $k \geq 1$ be an integer. The following statements are equivalent:*

(1) $H_i(T) = 0$ for $i = 2, \dots, k + 1$
(2) $d_1^{1,1}$ is an isomorphism, λ_i^S is an isomorphism for $i = 2, \dots, k$, and λ_{k+1}^S is surjective.

Proof. (1) $\implies$ (2): We first establish that $d_1^{1,1} : S \otimes_R W \to H_1(E)$ is injective (and is thus an isomorphism of S-modules). Recall that the terms $E_{p,q}^0$ are non-zero only for (p, q) satisfying $q \geq p \geq 0$. Thus $E_{2,1}^0 = 0$, and so $E_{1,1}^2 = \operatorname{Ker} d_1^{1,1}$. Moreover, $E_{1,1}^2 = E_{1,1}^\infty$ is (isomorphic to) a subquotient of $H_2(T)$, so that $d_1^{1,1}$ is injective, as desired.

We now focus on the properties of the maps λ_i described in (2). In addition, we show that the following condition holds:

(3) $E_{0,k+1}^2 = 0$ and $E_{p,q}^2 = 0$ for all $(p, q) \neq (0, 0)$ with $0 \leq p \leq q \leq p + k - 1$.

We will establish (2) and (3) by induction on k.

Suppose $k = 1$. We begin with (3) and show that $E^2_{q,q} = 0$ for all $q \geq 1$. Let C be the Cartan construction on the free S-module $H_1(E)$ and let D^n denote the strand $C_{*,n}$ (see Construction 1.4). Let Γ_i denote $\Gamma_i^S(W \otimes_R S)$. For each $q \geq 1$, we have morphisms relating the row E^1_{*q} of the E^1 page of the spectral sequence to the strand D^q:

$$\begin{tikzcd}
E^1_{*q}: & \cdots & H_2(E)\otimes_S \Gamma_{q-2} \arrow[l] & H_1(E)\otimes_S \Gamma_{q-1} \arrow[l] & \Gamma_q \arrow[l] & 0 \arrow[l] \\
D^q: & \cdots & \wedge_2^S H_1(E)\otimes_S \Gamma_{q-2} \arrow[l] \arrow[u, "\lambda_2^S\otimes\Gamma_{q-2}"] & H_1(E)\otimes_S \Gamma_{q-1} \arrow[l] \arrow[u, "\lambda_1^S\otimes\Gamma_{q-1}", "="'] & \Gamma_q \arrow[l] \arrow[u, "="'] & 0 \arrow[l]
\end{tikzcd} \tag{2.4}$$

From this diagram we conclude that $E^2_{q,q} = H_q(D^q)$. But Remark 1.7 yields that $H_q(D^q) = 0$ for all $q \geq 1$, and so $E^2_{q,q} = 0$ for all $q \geq 1$, as desired. It remains to show that $E^2_{0,2} = 0$. We have $E^2_{0,2} = E^\infty_{0,2}$; this is (isomorphic to) a subquotient of $H_2(T)$, so that $E^2_{0,2} = 0$.

For (2) note that λ_1^S is the identity map on $H_1(E)$; we now show that λ_2^S is surjective. Note that $E^1_{-1,2} = 0$, and so $E^2_{0,2} = \operatorname{Coker} d_1^{1,2}$. But $E^2_{0,2} = 0$, and so $d_1^{1,2}$ is surjective. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
E^1_{*,2}: & 0 & H_2(E) \arrow[l] & H_1(E)\otimes_S \Gamma_1 \arrow[l, "d_1^{1,2}"'] \\
D^2_*: & 0 & \wedge_2^S H_1(E) \arrow[l] \arrow[u, "\lambda_2^S"'] & \wedge_1^S H_1(E)\otimes_S \Gamma_1 \arrow[l] \arrow[u, "\lambda_1^S\otimes\Gamma_1"', "\cong"]
\end{tikzcd}$$

Thus λ_2^S is surjective, as desired.

Suppose now that $k \geq 2$. By construction and by induction $E^2_{1,k} = E^\infty_{1,k}$; this is (isomorphic to) a subquotient of $H_{k+1}(T)$, and so $E^2_{1,k} = 0$. Similarly, $E^2_{0,k} = 0$. These equalities yield the following commutative diagram with exact rows:

$$\begin{tikzcd}
E^1_{*k}: & 0 & H_k(E) \arrow[l] & H_{k-1}(E)\otimes_S \Gamma_1 \arrow[l] & H_{k-2}(E)\otimes_S \Gamma_2 \arrow[l] \\
D^k: & 0 & \wedge_k^S H_1(E) \arrow[l] \arrow[u, "\lambda_k^S"'] & \wedge_{k-1}^S H_1(E)\otimes_S \Gamma_1 \arrow[l] \arrow[u, "\lambda_{k-1}^S\otimes\Gamma_1"', "\cong"] & \wedge_{k-2}^S H_1(E)\otimes_S \Gamma_{k-2} \arrow[l] \arrow[u, "\lambda_{k-2}^S\otimes\Gamma_2"', "\cong"]
\end{tikzcd} \tag{2.5}$$

An application of the four lemma now gives that λ_k^S is an isomorphism.

To establish (3) it remains to show that $E^2_{pq} = 0$ for all $(p,q) = (q-k+1, q)$, where $q \geq k+1$. For each such q, we have the following commutative diagram:

$$\begin{tikzcd}
E^1_{*q}: & H_k(E)\otimes_S \Gamma_{q-k} & H_{k-1}(E)\otimes_S \Gamma_{q-k+1} \arrow[l] & H_{k-2}(E)\otimes \Gamma_{q-k+2} \arrow[l] \\
D^q: & \wedge_k^S H_1(E)\otimes_S \Gamma_{q-k} \arrow[u, "\lambda_k^S\otimes\Gamma_{q-k}"', "\cong"] & \wedge_{k-1}^S H_1(E)\otimes_S \Gamma_{q-k+1} \arrow[l] \arrow[u, "\lambda_{k-1}^S\otimes\Gamma_{q-k+1}"', "\cong"] & \wedge_{k-2}^S H_1(E)\otimes \Gamma_{q-k+2} \arrow[l] \arrow[u, "\lambda_{k-2}^S\otimes\Gamma_{q-k+2}"', "\cong"]
\end{tikzcd} \tag{2.6}$$

Hence we have an isomorphism $E^2_{q-k+1,q} \cong H_{q-k+1}(D^q)$. Noting that $q-k+1 \geq 2$, Remark 1.7 yields $H_{q-k+1}(D^q) = 0$, and so $E^2_{q-k+1,q} = 0$ for each $q \geq k+1$.

For (2), it remains to show that λ^S_{k+1} is surjective. As $E^1_{-1,k+1} = 0$, we have $\operatorname{Coker} d_1^{1,k+1} = E^2_{0,k+1}$. But $E^2_{0,k+1} = E^\infty_{0,k+1}$; this is (isomorphic to) a subquotient of $H_{k+1}(T)$, so that $E^2_{0,k+1} = 0$. This in turn yields that $d_1^{1,k+1}$ is surjective. From this, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
E^1_{*,k+1}: & 0 & H_{k+1}(E) \arrow[l] & H_k(E)\otimes_S \Gamma_1 \arrow[l, "d_1^{1,k+1}"'] \\
D^{k+1}_*: & 0 & \wedge^S_{k+1} H_1(E) \arrow[l] \arrow[u, "\lambda^S_{k+1}"'] & \wedge^S_k H_1(E)\otimes_S \Gamma_1 \arrow[l] \arrow[u, "\cong"', "\lambda^S_k\otimes \Gamma_1"]
\end{tikzcd} \tag{2.7}$$

From this, we conclude that λ^S_{k+1} is surjective.

(2) $\implies$ (1): As above, let C denote the Cartan construction on the free S-module $H_1(E)$. First, we have that $E^2_{p,q} = 0$ for $(p,q) = (0,k)$ and for all $(p,q) \neq (0,0)$ with $0 \le p \le q \le p+k-1$. Indeed, by utilizing commutative diagrams analogous to (2.4), (2.5), and (2.6), we have that $E^2_{p,q}$ is isomorphic to $H_p(D^q)$ for $(p,q) = (0,k)$ and for all $(p,q) \neq (0,0)$ with $0 \le p \le q \le p+k-1$. Consequently, Remark 1.7 yields that $E^2_{p,q} = 0$ for all such (p,q). Second, noting that λ^S_{k+1} is surjective, we see from a diagram analogous to (2.7) that $E^2_{0,k+1} = 0$ as well.

In particular, this vanishing of E^2 in this region yields that $E^\infty_{p,q} = E^2_{p,q} = 0$ for all (p,q) satisfying $0 < p+q \le k+1$. For each such (p,q) we have a finite filtration

$$0 = F_{-1}H_{p+q} \subseteq F_0 H_{p+q} \subseteq \cdots \subseteq F_{p+q}H_{p+q} = H_{p+q}(T). \tag{2.8}$$

Each quotient of consecutive terms has the form

$$\frac{F_p H_{p+q}}{F_{p-1}H_{p+q}} \cong E^\infty_{p,q} = 0.$$

We therefore conclude that each containment in (2.8) is an equality, and hence $H_{p+q}(T) = 0$ for all $0 < p+q \le k+1$, so that $H_i(T) = 0$ for all $i = 2,\dots,k+1$. □

With Proposition 2.3 in hand, we are now prepared to prove Theorem 2.1. Note that the result of Blanco, Majadas, and Rodicio ([10, Theorem 1]) establishes the equivalence (1) $\iff$ (2).

Proof of Theorem 2.1. (1) $\implies$ (2) was established by Tate [20], and (2) $\implies$ (3) is clear.

(3) $\implies$ (1): By Proposition 2.3, $H_1(E)$ is free as an S-module via $d_1^{1,1} : S\otimes_R W \to H_1(E)$, so that $H_1(E)$ has the desired basis. Moreover, λ^S_i is an isomorphism for $i = 1,2,\dots,b+1$. As $H_{b+1}(E) = 0$, we have that $\wedge^S_{b+1} H_1(E) = 0$, and so $\operatorname{rank}_S H_1(E) \le b$. Then for each $i > b+1$ we have the equality $\wedge^S_i H_1(E) = 0$ and λ^S_i is an isomorphism (of zero modules). □

A result of Kustin, Şega, and Vraciu ([16, Lemma 1.7]) provides (in the local case) an analogous classification for two-generated quasi-complete intersection ideals in terms of the vanishing of the homology of the Tate construction and a double annihilator condition.

The following construction provides the framework for the double annihilator condition.

Construction 2.9 ([16, 1.4]). Fix a basis $v_1,\dots,v_n$ of E_1 with $\partial^E(v_i) = f_i$. Suppose that $\boldsymbol{z} = \{z_1,\dots,z_n\}$ is a set of degree one cycles of E such that the homology classes $\{\operatorname{cls}(z_i)\}$ minimally generate $H_1(E)$. Then there exist $\{a_{ij} : i,j \in [1,n]\} \subset R$ such that

$$z_i = \sum_{j=1}^n a_{ij} v_j.$$

Let A denote the matrix (a_{ij}) and set $\Delta = \det A$. Then the map $\lambda_n^S : \wedge_n^S H_1(E) \to H_n(E)$ is given by

$$\operatorname{cls}(z_1) \wedge \cdots \wedge \operatorname{cls}(z_n) \mapsto \Delta v_1 \cdots v_n.$$

The equality $\nu_R(H_1(E)) = n$ holds whenever I is a quasi-complete intersection with $\operatorname{depth}(I, R) = 0$; see [5, 1.2].

Lemma 2.10 ([16, Lemma 1.7])**.** *Suppose $\nu_R(I) = 2$ and $\operatorname{depth}(I, R) = 0$. Then the following statements are equivalent:*

(1) I is a quasi-complete intersection.
(2) $H_2(T) = 0$, $(0 :_R I) = (\Delta)$, and $(0 :_R \Delta) = I$, where Δ is as defined in Construction 2.9.

Remark 2.11. Suppose $I = (f)$ is principal and there exists $g \in R$ with $(0 :_R f) = (g)$. If $H_2(T(f; g)) = 0$, then I is a quasi-complete intersection. Indeed, the Tate construction $T(f; g)$ has the following form:

$$0 \longleftarrow R \xleftarrow{\cdot f} R \xleftarrow{\cdot g} R \xleftarrow{\cdot f} \cdots$$

The hypothesis that $H_2(T(f; g)) = 0$ yields that $(0 :_R f) = (g)$, and consequently $H_i(T(f; g)) = 0$ for all $i > 0$ and I is a quasi-complete intersection.[1]

The following result illustrates another case in which the vanishing of $H_2(T)$ is sufficient to detect the quasi-complete intersection property of I.

Proposition 2.12. *Suppose $I = (f)$ and $\bigcap_{i \geq 1}(f^i) = (0)$. If $H_2(T) = 0$, then I is a quasi-complete intersection.*

Proof. Let E denote the Koszul complex $K(f; R)$. Suppose $\boldsymbol{z}$ is a set of non-zero elements of R such that $(\boldsymbol{z}) = \operatorname{ann}_R(f) = H_1(E)$. Let T denote the Tate construction $T(f; \boldsymbol{z}) = E\langle \boldsymbol{w} \mid \partial w_z = z\rangle$. In this context the differentials $\partial_2^T : W \to R$ and $\partial_3^T : W \to W$ are defined on basis elements by $\partial_2^T(w_z) = z$ and $\partial_3^T(w_z) = f w_z$.

It will suffice to show that $\boldsymbol{z} = \{z\}$. Suppose not, and pick distinct generators z, z' in $\boldsymbol{z}$. We will show by induction that $z, z' \in (f^i)$ for all $i \geq 1$.

Note that $Z_2(T)$ contains the non-zero cycle $\zeta = z' w_z - z w_{z'}$. As $H_2(T) = 0$ we have that ζ is a boundary. Thus $z' = r' f$ and $z = r f$ for $r, r' \in R$, so that $z, z' \in (f)$.

Suppose now that $z, z' \in (f^i)$ for some $i \geq 1$. Then there exists $s, s' \in R$ with $z = s f^i$ and $z' = s' f^i$. Then $Z_2(T)$ contains the non-zero cycle $s w_{z'} - s' w_z$, so that there exists $t, t' \in R$ with $s = t f$ and $s' = t' f$. Thus $z = t f f^i$ and $z' = t' f f^i$, so that $z, z' \in (f^{i+1})$, completing the induction.

Therefore $z = 0 = z'$, a contradiction.

□

3. Vanishing of homology of DG algebras

In Theorem 2.1, we see a situation in which a band of vanishing of $H_*(T)$ implies that $H_i(T) = 0$ for all $i > 0$. In this section we prove the following result, which continues this theme.

Theorem 3.1. *Set $b = \max\{i : H_i(E) \neq 0\}$, and suppose $\boldsymbol{z} = \{z_1, \ldots, z_b\}$ is a set of cycles whose homology classes generate $H_1(E)$. Let T be the Tate construction on $\boldsymbol{f}$ and $\boldsymbol{z}$. If there exists an integer $q \geq 2$ with $H_i(T) = 0$ for $i \in \{q, \ldots, q + b\}$, then I is a quasi-complete intersection.*

The hypothesis that $H_1(E)$ can be generated by b elements means that the size of $H_1(E)$ is compatible with I being a quasi-complete intersection in the following sense:

Remark 3.2. If I is a quasi-complete intersection ideal, then $\operatorname{rank}_S H_1(E) = b$. Indeed, the isomorphism λ_*^S yields the equality $b = \max\{i : \wedge_i^S H_1(E) \neq 0\}$.

[1] In this context, the pair (f, g) is an *exact pair* in the sense of Kiełpiński, Simson, and Tyc [15, Definition 1.1].

We now develop conditions, expressed as a band of vanishing of homology, under which an extension formed by the adjunction of variables of degree two exhibits eventually periodic or eventually vanishing homology. We adopt the notation of [4, §6]. For integers $i \leq j$, let $[i,j]$ denote $\{i, i+1, \dots, j\}$.

Lemma 3.3. *Let A denote a DG R-algebra. Suppose that $\{z_1, \dots, z_m\}$ is a set degree one cycles of A. Put $A_0 = A$ and for $1 \leq j \leq m$ put $A_j = A_{j-1}\langle w_j \mid \partial w_j = z_j\rangle$. Let q and b be integers.*

(1) Suppose $H_i(A_m) = 0$ for all $i \in [q, q+m]$. Then for each j we have $H_i(A_j) = 0$ for all $i \in [q+m-j, q+m]$.

Suppose further that $H_i(A) = 0$ for all $i > b$.

(2) $H_i(A_1) \cong H_{i+2}(A_1)$ for all $i \geq b$, i.e., $H_*(A_1)$ is periodic of period 2 beginning in degree b.
(3) If $q \geq b+1-m$ and $H_i(A_m) = 0$ for all $i \in [q, q+m]$, then $H_i(A_m) = 0$ for all $i \geq b+1-m$.

Proof. For (1), by induction we may assume $m = 1$. Then $H_i(A_1) = 0$ for $i \in \{q, q+1\}$. The equality $H_{q+1}(A) = 0$ now follows from immediately from the following portion of long exact sequence in homology associated with Tate's exact homology triangle [4, Remark 6.1.6]:

$$
\begin{array}{c}
\cdots \longrightarrow H_{q+1}(A_1) \longrightarrow \\
H_{q+2}(A) \longrightarrow H_{q+2}(A_1) \longrightarrow H_q(A_1) \longrightarrow \\
H_{q+1}(A) \longrightarrow H_{q+1}(A_1) \longrightarrow \cdots
\end{array} \tag{3.4}
$$

The result of (2) also follows from [4, Remark 6.1.6]: For each $i \geq b$ we have an isomorphism $H_{i+2}(A_1) \cong H_i(A_1)$, so that $H_*(A_1)$ is eventually periodic of period 2. The extremal case occurs when $i = b$, namely $H_{b+2}(A_1) \cong H_b(A_1)$, so that the periodicity begins in the desired position.

For (3), we proceed by induction on m. Consider the case $m = 1$. By (2), the vanishing of $H_i(A)$ for $i > b$ yields that $H_*(A_1)$ is periodic of period 2 beginning in degree b. By hypothesis, $H_i(A_1) = 0$ for $i \in \{q, q+1\}$. As $q \geq b$, we have that one representative from each of the two isomorphism classes of $H_{\geq b}(A_1)$ vanishes, so that $H_i(A_1) = 0$ for all $i \geq b$.

Suppose now that for each $1 \leq a < m$ the statement holds for the adjunction of a variables of degree two, and that $H_i(A_m) = 0$ for all $i \in [q, q+m]$. By (1), $H_i(A_{m-1}) = 0$ for all $i \in [q+1, q+m]$. By induction, we have that $H_i(A_{m-1}) = 0$ for all $i \geq b+1-(m-1)$, so that (2) yields that $H_*(A_m)$ is periodic of period 2 starting in degree $b+1-m$. As $q \geq b+1-m$, we have that (at least) one representative from each of the two isomorphism classes of $H_{\geq b+1-m}(A_m)$ vanishes, which completes the proof. □

Remark 3.5. The vanishing guaranteed by Lemma 3.3 begins at a position independent of the location of the band of vanishing. In particular, this yields the following: If $H_i(B) = 0$ for all $i \gg 0$, then $H_i(B) = 0$ for all $i \geq b+1-m$.

Remark 3.6. In the more general case where the extension B is formed by the adjunction of variables in a single *arbitrary* even degree or *differing* even degrees, one can obtain a description of a region of vanishing which implies the eventual vanishing of the homology of such an extension.

Proof of Theorem 3.1. Note that $T = E\langle w_1, \dots, w_b \mid \partial w_i = z_i\rangle$, where $|z_i| = 1$. Lemma 3.3 (3) now yields that $H_i(T)$ for all $i \geq 1$. Thus T is acyclic, and I is a quasi-complete intersection. □

4. Characterizing complete intersections

In this section, $(R, \mathfrak{m}, k)$ is a (Noetherian) local ring.

Definition 4.1. We say that R is a *complete intersection* if its $\mathfrak{m}$-adic completion $\widehat{R}$ can be written as a quotient of a (complete) regular local ring by a regular sequence.

A result of Assmus [1, Theorem 2.7] yields that R is a complete intersection if and only if $\mathfrak{m}$ is a quasi-complete intersection ideal. Assmus' result does not use the quasi-complete intersection terminology: The condition is stated as "$H(E)$ is the exterior algebra on $H_1(E)$". As Avramov, Henriques, and Şega [5, §1] note, the existence of *some* isomorphism of graded S-algebras

$$\lambda : H(E) \xrightarrow{\cong} \wedge_*^S H_1(E)$$

guarantees the quasi-complete intersection property.

Let T denote Tate construction on $\mathfrak{m}$; see Remark 1.2. In this section, we study complete intersection rings by applying the results of Sections 2 and 3. We show that, compared to the non-maximal case, the quasi-complete intersection property of $\mathfrak{m}$ (and hence the complete intersection property of R) can be detected from a smaller band of vanishing of $H_*(T)$. Hereafter, the size of a minimal generating set of an R-module M is denoted $\nu_R(M)$.

We begin by outlining a construction which will allow us to relate a Tate construction over local ring to a Tate construction over a quotient.

Construction 4.2. *The Tate construction on R.* Assume that R is complete. There exits a regular local ring $(Q, \mathfrak{n}, k)$ and an ideal $J \subset \mathfrak{n}^2$ such that $R = Q/J$. Furthermore, $b = \nu_Q(J)$; see, for example, [1, pp 196-197]. Select a maximal Q-sequence $a_1, \dots, a_h$ in J so that the images $\{\overline{a}_i\}$ in $J/\mathfrak{n}J$ are linearly independent over $Q/\mathfrak{n}$; we may extend the sequence to a minimal generating set $a_1, \dots, a_b$ of J. Put $J' = (a_1, \dots, a_h)$ and let $(Q', \mathfrak{n}')$ denote $(Q/J', \mathfrak{n}/J')$.

Let K denote the Koszul complex on a minimal generating set of $\mathfrak{n}$ and let E' denote the Koszul complex on a minimal generating set of $\mathfrak{n}'$. As before, $h = \nu_{Q'}(H_1(E'))$. Let $\boldsymbol{z'} = z'_1, \dots, z'_h$ denote the set of cycles given by the construction in [1, pp 196-197]; their homology classes form a minimal generating set for $H_1(E')$. Moreover, letting $z_1, \dots z_h$ denote their images in $E = E' \otimes_{Q'} R$, the same construction yields that these images extend to a set of cycles $\boldsymbol{z} = z_1, \dots, z_b$ whose homology classes form a minimal generating set of $H_1(E)$.

Let F' denote the Tate construction on E' and $\boldsymbol{z'}$, so that $F' = E'\langle w_1, \dots, w_h \mid \partial w_i = z'_i\rangle$. Set $F = F' \otimes_{Q'} R = E\langle w_1, \dots, w_h \mid \partial w_i = z_i\rangle$. By construction, Q' is a complete intersection; a result of Assmus ([1, Theorem 2.7]) yields that F' is a minimal Q'-free resolution of k, and thus $\operatorname{Tor}_i^{Q'}(R, k) = H_i(F)$. Let T denote that Tate construction on E and $\boldsymbol{z}$. Then $T = F\langle w_{h+1}, \dots, w_b \mid \partial w_i = z_i\rangle$.

Let π denote the natural surjection $Q' \to R$; we note a connection between $\operatorname{Ker} \pi$ and $\operatorname{pd}_{Q'} R$.

Remark 4.3. By Construction 4.2, $\operatorname{Ker} \pi$ contains only zerodivisors. A result of Auslander and Buchsbaum [2, Proposition 6.2] now yields the implication $\operatorname{pd}_{Q'} R < \infty \implies (0 :_{Q'} R) = 0$.

The following result (Theorem B) is the improvement of Theorem 3.1.

Theorem 4.4. *Suppose there exists an integer $q \geq 2$ such that $H_i(T) = 0$ for $i = [q, q + b - 1]$. Then R is a complete intersection.*

Proof. Here we follow the strategy of Gulliksen [11]. Without loss of generality, we may assume that R is complete. Recall the notation of Construction 4.2. Let $\pi : Q' \to R$ be the natural surjection. We will show that $\operatorname{Ker} \pi = 0$; by Remark 4.3 it will be enough to show that $\operatorname{pd}_{Q'} R < \infty$.

Recall that $\operatorname{Tor}_i^{Q'}(R, k) = H_i(F)$. By hypothesis, there exists an integer $q \geq 2$ such that $H_i(T) = 0$ for $i = [q, q+b-1]$. Noting that we have obtained T from F by adjoining at most $b-1$ variables of degree two, Lemma 3.3 (1) yields that $H_{q+b-1}(F) = 0$. This implies that $\operatorname{Tor}_{q+b-1}^{Q'}(R, k) = 0$ for some $q \geq 2$. Hence, $\operatorname{pd}_{Q'} R < \infty$, completing the proof. □

Let $R\langle X\rangle$ denote an acyclic closure of k over R and order the variables X such that $|x_i| \leq |x_j|$ for $i < j$; see [4, Construction 6.3.1]. Fix an integer p and let Y denote the extension[2] $R\langle x_i : i \leq p\rangle$.

We note that the following result appears implicitly in work of Gulliksen [11]:

Proposition 4.5. *Let F be as in Construction 4.2 and suppose that $F \subseteq Y$. If $H_i(Y) = 0$ for all $i \gg 0$, then R is a complete intersection.*

Proof. The DG-algebra Y satisfies the conditions of [11, Lemma 1]. Now $H_i(Y) = 0$ for all $i \gg 0$ and Y is obtained from F by an adjunction of (finitely many) variables, so a repeated application of [11, Lemma 2] yields $H_i(F) = 0$ for all $i \gg 0$. But $H_i(F) = \operatorname{Tor}_i^{Q'}(R, k)$, so that $\operatorname{pd}_{Q'} R < \infty$. Consequently, Remark 4.3 yields that R is a complete intersection. $\square$

In particular, the eventual vanishing of $H_*(T)$ is equivalent to the complete intersection property of R (i.e, the quasi-complete intersection property of $\mathfrak{m}$).

Assmus [1, Theorem 2.7] establishes that the complete intersection property of R is equivalent to the vanishing of $H_2(T)$. We now develop the tools needed to extend on this result to show that the vanishing of $H_3(T)$ or $H_4(T)$ also detects the complete intersection property.

The following lemma highlights two situations in which the adjunction of variables to annihilate a non-zero homology class preserves the vanishing of homology in a higher degree.

Lemma 4.6. *Let A be a DG R-algebra and assume that $H_0(A) = k$. Let i be an integer, and suppose that $H_i(A) \neq 0$. Let z be a cycle representing a non-zero homology class in $H_i(A)$ and set $B = A\langle w \mid \partial w = z\rangle$.*

(1) If $i \geq 2$ is even and $H_1(A) = 0 = H_{i+2}(A) = 0$, then $H_1(B) = 0 = H_{i+2}(B) = 0$.
(2) If $H_{i+1}(A) = 0$, then $H_{i+1}(B) = 0$.

Proof. For (1), the equality $H_1(B) = 0$ is clear, and the equality $H_{i+2}(B) = 0$ follows immediately from a portion of the exact sequence from [4, Remark 6.1.5]:

$$\begin{array}{ccccccc} \cdots \longrightarrow & H_{i+2}(A) & \longrightarrow H_{i+2}(B) \xrightarrow{H_{i+1}(\vartheta)} & H_1(A) & \longrightarrow \cdots \\ & \| & & \| & \\ & 0 & & 0 & \end{array}$$

Let ζ denote $\operatorname{cls}(z)$. For (2), suppose first that i is even.

We consider the following portion of exact sequence in homology of [4, Remark 6.1.5]:

$$\begin{array}{ccccccc} \cdots \longrightarrow & H_{i+1}(A) & \longrightarrow H_{i+1}(B) \xrightarrow{H_i(\vartheta)} & H_0(A) & \xrightarrow{\cdot\zeta} H_i(A) \longrightarrow \cdots \\ & \| & & \| & \\ & 0 & & k & \end{array}$$

Multiplication by ζ is injective on $H_0(A)$, so that $H_{i+1}(B) = 0$, as desired.

In the case where i is odd, the relevant portion of the exact sequence in homology from [4, Remark 6.1.6] is the following:

$$\begin{array}{ccccccc} H_{i+1}(A) & \longrightarrow H_{i+1}(B) \xrightarrow{H_{i+1}(\vartheta)} & H_0(B) & \xrightarrow{\eth_{i+1}} H_i(A) \xrightarrow{H_i(\iota)} H_i(B) \longrightarrow & H_{-1}(B) \\ \| & & \| & & \| \\ 0 & & k & & 0 \end{array}$$

By construction $H_i(\iota)(\zeta) = 0$, and so $H_i(\iota)$ is not injective. Thus $\eth_{i+1}$ is not the zero map and so $\eth_{i+1}$ is injective. Thus $H_{i+1}(B) = 0$. $\square$

[2]An example of such an extension is a *partial acyclic closure* $R\langle X_{\leq n}\rangle$.

In the next result (Theorem C, condition (1)), we make use of the *deviations* $\varepsilon_n(R)$ of R, for which we use [4, §7] as a reference. Let T denote the Tate construction on E; see Remark 1.2.

Theorem 4.7. *If $H_i(T) = 0$ for some $i = 3$ or 4, then R is a complete intersection.*

Proof. We may assume that R is not a complete intersection, so that $H_2(T) \neq 0$.

If $H_3(T) \neq 0$, then we apply Lemma 4.6 (2) and adjoin variables of degree three to obtain a partial acyclic closure B of k with $H_i(B) = 0$ for $i \in \{1, 2, 3\}$. This yields $\varepsilon_4(R) = 0$, so that by a result of Gulliksen [12, Theorem 3.5.1], R is a complete intersection, a contradiction.[3]

Suppose now that $H_4(T) = 0$. We adjoin variables of degrees 3 and 4; applying Lemma 4.6 (1) and (2), we obtain a partial acyclic closure V of k with $H_i(V) = 0$ for $i = 1, 2, 3, 4$, so that $\varepsilon_5(R) = 0$. Now Halperin [13, Theorem B] gives that R is a complete intersection, a contradiction. □

5. Rigidity of the Tate construction

In this section $(R, \mathfrak{m}, k)$ is a local ring and let T denote the Tate construction on $\mathfrak{m}$.

Previous work ([1, Theorem 2.7]) and the work of this paper (Theorem 4.7) suggest the following question:

Question 5.1. Does the implication

$$H_i(T) = 0 \text{ for some } i \geq 0 \implies R \text{ is a complete intersection}$$

hold for every local ring R?

Suppose that $\varphi : Q \to R$ is a surjective homomorphism of local rings and M is a finite R-module. Recall the *Poincaré series* of M over R:

$$P^R_M(t) = \sum_{n=0}^{\infty} \beta^R_n(t) t^n \in Z[[t]].$$

The following result relates the Betti numbers of M over R and Q.

Proposition 5.2. [4, Proposition 3.3.2] *Then there is a coefficientwise inequality of formal power series*

$$P^R_M(t) \preccurlyeq \frac{P^Q_M(t)}{1 - t(P^Q_R(t) - 1)}. \tag{5.3}$$

We present a class of rings for which Question 5.1 holds. This class is defined in terms of *Golod homomorphisms*, for which we use [3, 4] as references.

Definition 5.4. [4, §3.3] A surjective homomorphism $\varphi : Q \to R$ is called a *Golod homomorphism* if equality holds in (5.3) for $M = k$.

Theorem 5.5. *Suppose that there exists a complete intersection ring Q and a Golod homomorphism $\varphi : Q \to \widehat{R}$. If $H_i(T) = 0$ for some $i \geq 5$, then R is a complete intersection.*

This is Theorem C, condition (2).

Proof. By [8, Proposition 5.13] we may assume that $\operatorname{depth}_Q(R) = 0$. We endeavor to show that $\operatorname{Ker} \varphi = 0$. By Remark 4.3 it is enough to show that $\operatorname{pd}_Q R < \infty$.

Let F' denote the Tate construction on $\mathfrak{n}$, and put $F = R \otimes_Q F'$. As Q is a complete intersection, we have that F' is a minimal Q-free resolution of k. Let A denote the trivial extension $k \ltimes H_{\geq 1}(F)$. Then [3, Theorem 2.3] yields that F and A are equivalent as DG-algebras.

Let $\boldsymbol{y}$ be a set of cycles of degree one whose homology classes form a minimal generating set of $H_1(F)$, and let X denote the Tate complex on A and $\boldsymbol{y}$. Then [12, Proposition 1.3.5] yields the equivalence $T \simeq X$. Thus, there exists an integer $i \geq 5$ with $H_i(X) = 0$.

[3] The indexing convention of the ε_n differs from that of Gulliksen and Levin [12]; ε_3 of [12] stands for ε_4 of [4].

As the differential on A is trivial, we observe that X exhibits a direct sum decomposition (cf. Remark 1.5):

$$X = \bigoplus_{j \geq 0} D^j,$$

where D^j is the complex

$$0 \longleftarrow H_j(F) \xleftarrow{\partial_1^{D_j}} H_{j-1}(F) \otimes \Gamma_1^k W \xleftarrow{\partial_2^{D_j}} \cdots \longleftarrow H_1(F) \otimes \Gamma_{j-1}^k W \xleftarrow{\partial_j^{D_j}} \Gamma_j^k W \longleftarrow 0$$

Consequently, we have a decomposition of the homology of X:

$$H_k(X) = \bigoplus_{i \geq 0} H_i(D^{k-i}) = \bigoplus_{i=0}^{k} H_i(D^{k-i}). \tag{5.6}$$

The equivalence $F \simeq A$ yields that $[H_{\geq 1}(F)]^2 = 0$, and so the differential $\partial_i^{D_j}$ is zero for each i in $\{1, 2, \dots, j-1\}$. In light of (5.6), this yields that $H_0(D^k) = H_k(F)$ for each $k \geq 2$, so that $H_k(X)$ contains $H_k(F)$ as a summand for each $k \geq 2$. As such, $H_i(F) = 0$, and so $\operatorname{Tor}_i^Q(R, k) = 0$. Therefore, $\operatorname{pd}_Q R < \infty$, and hence R is a complete intersection. □

Remark 5.7. The hypotheses of Theorem 5.5 are satisfied in the following situations:

1. R is a Golod ring,
2. R is Gorenstein and $\operatorname{embdim} R = 4$; see [14, Theorem B],
3. $\operatorname{codepth} R \leq 3$; see [8, Proposition 6.1],
4. $\mathfrak{m}$ has a *Conca generator* (i.e., there exists $x \in \mathfrak{m}$ such that $x^2 = 0$ and $\mathfrak{m}^2 = x\mathfrak{m}$); see [7, Theorem 1.4].
5. R is a *compressed Gorenstein ring* of socle degree s and embedding dimension e for $2 \leq s \neq 3$ and $e > 1$; see [19, Theorem 5.1].

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Department of Mathematics, Gonzaga University, Spokane, WA 99258-0102, USA
E-mail address: lutzj@gonzaga.edu