Chapter 11 – Summary and Review (draft: 2019/02/07-13:45:51)

11.1 Vectors in the plane

Summary of topics and terminology:

- Construct a vector from two points
- Distance between points
- Magnitude/length of vector
- Scalar multiplication
- Zero vector
- Rules of vector arithmetic, properties of vectors
- Adding and subtracting vectors graphically
- Unit vector

Example problems:

1. Given points $P(1, 3)$ and $Q(-2, 1)$: find the distance between them, find vector $\vec{PQ}$ and its magnitude.
2. Given $\vec{u} = \langle 0, 1\rangle$ and $\vec{v} = \langle 5, -1\rangle$, find $\vec{u} + 3\vec{v}$. Sketch all vectors in the plane.

11.2 Vectors in 3 dimensions

Summary of topics and terminology:

- Plotting points and vectors
- Distance formula and vector magnitude
- Angle bracket and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ representation of vectors
- Right hand rule for orientation of 3D axes
- Planes that are parallel to coordinate planes
- Spheres and (solid) balls
- Boxes and (solid) blocks
- Vector operations, addition, subtraction, scalar multiplication, etc.

Example problems:

1. Identify the center and radius of the sphere $x^2 - 2x + y^2 + 8y + z^2 + 2z = 1$.
2. Describe the 3D region given by $x^2 + y^2 + z^2 \leq 9$.
3. Describe the 3D region given by $0 \leq x \leq 5$, $0 \leq y \leq 2$, $0 \leq z \leq 1$.
4. Explain what $y = -3$ represents.

5. Describe the region $9 \leq x^2 + y^2 + z^2 \leq 16$, $z \geq 0$.

6. Describe $0 \leq z \leq 1$.

11.3 Dot products

Summary of topics and terminology:

- Calculate dot product of vectors
- Angle between vectors
- Scalar and vector orthogonal projections
- You should be refreshing your right triangle trigonometry as well

Example problems:

1. Consider vectors $\mathbf{u} = \langle 2, 1, 3 \rangle$, and $\mathbf{v} = \mathbf{i} - 3\mathbf{k}$. Find their dot product, the angle between them, and the scalar and vector projections of $\mathbf{u}$ onto $\mathbf{v}$.

2. Find all t that make vector $\mathbf{v} = \langle 2t, -3, 6t \rangle$ orthogonal to $\mathbf{u} = \langle -1, 1, 2 \rangle$.

11.4 Cross products

Summary of topics and terminology:

- Calculate cross product of vectors
- Area of parallelogram
- Right hand rule for cross products.

Example problems:

1. Consider vectors $\mathbf{u} = \langle 2, 1, 3 \rangle$, and $\mathbf{v} = \mathbf{i} - 3\mathbf{k}$. Find their cross product, and the area of the parallelogram they create.

2. Find the area of triangle created by points $P(0, 1, 1)$, $Q(-1, 0, 2)$, and $R(1, 1, 0)$.

3. Find 3 vectors that are orthogonal to $\langle 1, 1, 1 \rangle$ and $\langle -1, 0, 2 \rangle$. (Hint: Vectors that point in the same direction but with different lengths are technically different vectors!)

11.5 Lines and curves in space

Summary of topics and terminology:

- Vector-valued function as 3 parametric equations that give s curve in 3D space
- Vector and parametric representation of lines
- Find line 3 ways: from two points, point and direction/parallel vector, point and two perpendicular vectors
- Equation of line segment
- Intersection of line and plane

Example problems:

1. Find the equation of the line that goes through points $P(0, 1, 2)$ and $Q(1, 0, 2)$.
2. Find the equation of the line that goes through point $(1, 2, 3)$ and is parallel to vector $\langle -1, 1, 2\rangle$.
3. Find the equation of the line that goes through point $(1, 2, 3)$ and is perpendicualr to vectors $\langle 1, 0, 1\rangle$ and $\langle 0, 1, 1\rangle$.
4. Find the intersection of the line $\mathbf{r}(t) = \langle t + 2, 3 - t, 2t\rangle$ with the plane $y = 2x - 1$.
5. Describe the curve given by $\mathbf{r}(t) = \langle \cos(t), \sin(t), -t\rangle$.
6. Describe the curve given by $\mathbf{r}(t) = \langle \sin(t), t, \cos(t)\rangle$.
7. Describe the curve given by $\mathbf{r}(t) = \langle \cos(t), 3 \sin(t), t\rangle$.

11.6 Calculus of vector-valued functions

Summary of topics and terminology:

- Take derivatives $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.
- Apply product rule to dot and cross products.
- Unit tangent vector.
- Indefinite and definite integrals of vector functions.

Example problems:

1. Find the unit tangent vector to $\mathbf{r}(t) = \langle \cos(t), 3 \sin(t), t\rangle$.
2. Integrate $\mathbf{r}(t)$ above from $t = 0$ to $t = 2\pi$. Hint: the result is a vector.

11.7 Motion in space

Summary of topics and terminology:

- Position, velocity, acceleration in 2D and 3D (vectors).
- Speed (scalar).
- If $|\mathbf{r}(t)|$ is constant, then motion is on a circle or sphere, and $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$. This means the velocity vector is orthogonal to the position vector. Note that the position vector goes from the origin to the point on the curve.

Example problems:

1. (2D) A projectile is fired from a cliff that is 20m above the ground below. It has initial speed 250 m/s and was fired at an angle of 20° above horizontal. Find the projectiles maximum height above the ground, how far it travels horizontally, and its total flight time.
2. (3D) Find the position, velocity, and acceleration of a projectile launched from the origin at an angle 30° from the positive x-axis (counter-clockwise towards the positive y-axis), and 45° above the xy-plane (towards the positive z axis) with initial speed 100 m/s. Assume that gravity is the only source of acceleration.

11.8 Length of curves

Summary of topics and terminology:

- Calculate arc length, $L = \int_a^b |\mathbf{r}'(t)|dt$.
- Find arc length function $s(t) = \int_0^t |\mathbf{r}'(u)|du$.
- $\frac{ds}{dt} = |\mathbf{r}'(t)|$ is the speed, so rate of change of arc length w.r.t. time is speed.
- Reparametrize $\mathbf{r}(t)$ in terms of arc length. If $s = f(t)$, then $f^{-1}(s) = t$ so $\mathbf{r}_\ell(s) = \mathbf{r}(f^{-1}(s))$.

Example problems:

1. Consider the curve given by $\mathbf{r}(t) = \langle 2t, 3\cos(t), 3\sin(t)\rangle$. Find the arc length for $0 \le t \le 2\pi$. Find the arc length function $s(t)$. Reparametrize the curve in terms of arc length. If $\mathbf{r}(t)$ described the motion of a particle in space, what is the speed of that particle?

11.9 Curvature and normal vectors

Summary of topics and terminology:

- Curvature: $\kappa(s) = \left|\frac{d\mathbf{T}}{ds}\right|$
- Curvature: $\kappa(t) = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$.
- Curvature: $\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$.
- Radius of curvature is $\frac{1}{\kappa}$. This is the radius of the circle that best fits the curve at a particular point.
- Unit tangent vector: $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.
- Unit normal vector: $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$.
- $\mathbf{T}\cdot\mathbf{N} = 0$ (orthogonal)
- Binormal vector (also a unit vector): $\mathbf{B} = \mathbf{T}\times\mathbf{N}$.
- Torsion (scalar): $\tau = -\frac{d\mathbf{T}}{ds}\cdot\mathbf{N} = \frac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{a}'}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}'\times\mathbf{r}''|^2}$
- Think of the $\mathbf{T},\mathbf{N},\mathbf{B}$ vectors as forming a 3D coordinate system that follows the curve along with $\mathbf{T}$ pointing in the forward direction, $\mathbf{N}$ pointing towards the inside of the curve (e.g. towards the center of the circle), and $\mathbf{B}$ defining the "up" direction relative the the place created from $\mathbf{T}$ and $\mathbf{N}$.
- The normal plane is created by $\mathbf{N}, B$. The curve is perpendicular to the normal plane, think of the curve as piercing the normal plane, like a pencil through a piece of paper.
- The osculating plane is created by $\mathbf{T}, N$. Think of the curve as "living" in the osculating plane. This is not totally accurate, but the way the path curves defines the osculating plane.

Example problems:

1. Consider the curve given by $\mathbf{r}(t) = \langle 2t, 3\cos(t), 3\sin(t)\rangle$. Find the $\mathbf{T},\mathbf{N},\mathbf{B}$ vectors. Calculate the curvature at $t = \pi$.