Chapter 12 – Summary and Review (draft: 2019/03/01-15:31:12)

12.1 Planes and surfaces

Summary of topics and terminology:

- Construct equation/formula for plane multiple ways (from 3 pts, from 1 pt & normal vector, from 1 pt & 2 vectors in plane, from 1pt and parallel plane, etc.)
- Angle between planes is angle between normal vectors
- Line of intersection of planes
- Cylinders (any curve that lives in any plane and extruded along any line not in that plane)
- Traces, be able to describe what they are and sketch them (2D conics, curves, and lines, i.e. parabolas, hyperbolas, ellipses, etc.)
- Quadric surfaces (especially be able to identify and sketch their traces)
- Find $x, y,$ and z intercepts of planes or other surfaces (plug in zero for other two variables to the the intercept of one, e.g. $y = z = 0$ and solve for x to get x-intercept)

Example problems:

1. Find normal vector to plane $4x - 2y + z = 1$ and find 3 points in that plane that are not on a line.
2. Find the angle between planes $x + 2z = 4$ and $3x + y - z = 6$.
3. Find equation of plane that is parallel to plane $x + 3y - z = 2$ and contains point $(2, 1, 3)$.
4. Find the plane with points $P(1, 3, 0)$, $Q(1, -2, 1)$, and $R(0, 2, 1)$.
5. Find the line of intersection of planes $x + y + 2z = 5$ and $x - y + z = 1$.
6. Describe the graph of $z = x^2$, sketch it.
7. Describe and plot traces for the graph of $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 1$. Identify the surface.
8. Describe and plot traces for the graph of $\frac{x^2}{4} + y^2 - \frac{z^2}{9} = 1$. Identify the surface.
9. Describe and plot traces for the graph of $\frac{x^2}{4} = y^2 - \frac{z^2}{9}$. Identify the surface.
10. Describe and plot traces for the graph of $\frac{x}{2} = y^2 - \frac{z^2}{9}$. Identify the surface.

12.2 Graphs and level curves

Summary of topics and terminology:

- $z = f(x, y)$ surface
- $T = f(x, y, z)$ hypersurface
- Level curves of $z = f(x, y)$ plotted in xy-plane
- Level sets of $T = f(x, y, z)$ plotted in $\mathbb{R}^3$.

- Domain of $z = f(x, y)$
- Contour plot (level curves) like a topo hiking map
- contours close together where surface is "steep", contours further apart where surface is "flatter".

Example problems:

1. Find and sketch domain of $f(x, y) = \sqrt{5 - 2x + y}$.
2. Find and sketch domain of $f(x, y) = \ln(4 - x^2 - y^2)$.
3. Find and sketch domain of $f(x, y, z) = z + \sqrt{9 - x^2 - y^2}$.
4. Sketch level curves (contours) $z = 0, 1, 2, 3$ for $z = \sqrt{x^2 - y}$.
5. Sketch level curves (contours) $z = 0.25, 0.5, 0.75, 1.0$ for $z = e^{-x^2-y^2}$.

12.3 Limits and continuity

Summary of topics and terminology:

- Show a limit does not exist by checking along different curves
- Evaluate $\lim_{(x,y)\to(0,0)} f(x, y)$ along $x = 0$, $y = 0$, $x = y$, general $y = mx$, could try $y = x^2$ or $y = x^n$ or others also.

Example problems:

1. Show limit does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x + y}{x + 5y}$
2. Show limit does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 - y^2}$

12.4 Partial derivatives

Summary of topics and terminology:

- Calculate first, second, and higher partial derivatives
- Estimate whether f_x and f_y at a specific point are positive, negative, or zero from a surface graph
- Estimate f_x and f_y numerically at a specific point from a contour/level curve plot. f_x or f_y both $\approx \frac{\text{change in elevation}}{\text{change in horizontal distance in the plane}}$

Example problems:

1. Calculate all first and second partial derivatives of $f(x, y) = x^2y - y^3 + x^5$.
2. Estimate f_x and f_y at the points $(1, 1)$, $(0, 2)$, $(1, 0)$, and $(-2, 2)$ using this contour plot: https://www.desmos.com/calculator/necenz46el (note that you will need to figure out what the z values are for the contours.)

3. Estimate f_x and f_y (at least whether they are positive, negative, or zero) at the points $(0,0)$, $(0,1)$, $(2,0)$, and $(-1,1)$ using this surface plot: https://www.math3d.org/J7awICn3

12.5 Chain rule

Summary of topics and terminology:

- Calculate derivatives of multivariable functions using chain rule
- Know when to use d derivative notation and when to use ∂ derivative notation. It depends on what level you are taking the derivative at, and whether or not there are multiple variables to take derivatives w.r.t. at that level. see example problems below.
- Illustrate the chain rule with a tree diagram.
- Implicit differentiation for x, y: for $F(x,y) = C$ (constant), we can think of y as implicitly a function of x and get $\frac{dy}{dx} = -\frac{F_x}{F_y}$.
- Implicit differentiation for x, y, z: for $F(x,y,z) = C$ (C a constant), we can think of z as implicitly a function of x and y and we get $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$.

Example problems:

1. $z = f(x,y)$, $x = x(t)$, $y = y(t)$. Write the general formula for $\frac{dz}{dt}$.
2. $z = f(x,y)$, $x = x(t,s)$, $y = y(t,s)$. Write the general formula for $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$.
3. $T = f(x,y,z)$, $x = x(t)$, $y = y(t)$, $z = z(t)$. Write the general formula for $\frac{dT}{dt}$.
4. $z = f(x,y)$, $x = x(t,s)$, $y = y(t,s)$, $s = s(u,v)$, $t = t(u,v)$. Write the general formula for $\frac{\partial z}{\partial u}$ $\frac{\partial z}{\partial v}$.
5. $z = x^2y - xy^3$, $x = s + t$, $y = s^2t$. Find $\frac{\partial z}{\partial t}$ and $\frac{\partial z}{\partial s}$.
6. Find dy/dx by implicit differentiation: $x^2y + xy^3 = 1$.
7. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ by implicit differentiation: $x^2yz + xy^3z^2 = xyz + z^4$.

12.6 Directional derivative

Summary of topics and terminology:

- Estimate directional derivative from a contour map
- Calculate directional derivative in direction of both unit and non-unit vectors.
- $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$ for $\mathbf{u}$ a unit vector.
- $D_{\mathbf{v}}f = \nabla f \cdot \frac{\mathbf{v}}{|\mathbf{v}|}$ for any vector $\mathbf{v}$.
- This works in both 2D and 3D.

Example problems:

1. Find the derivative of $f(x,y) = x^2 + xy - y^2$ in the direction of $\langle 1, 2\rangle$ at the point $(1,-1)$.
2. Find the derivative of $g(x,y,z) = xyz$ in the direction of $\langle 1, 2, -1\rangle$ at the point $(3,2,1)$.

3. Estimate the derivative of f at the points $(0,0)$, $(0,1)$, $(2,0)$, and $(-1,1)$ using this contour plot: https://www.desmos.com/calculator/necenz46el (note that you will need to figure out what the z values are for the contours.)

12.7 Tangent planes and linear approximation

Summary of topics and terminology:

- Find tangent plane to surface $z = f(x, y)$.
- Tangent plane is $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$.
- Linear approximation to $z = f(x, y)$ is
  $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.
- Differential $dz = f_x dx + f_y dy$.
- Use differential to approximate a small change in z for small changes in x and y,
  $\Delta z \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$.
- The **e**xact change in z is $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$
- Higher dimensional differentials: e.g.
  for $T = f(x, y, z)$, then $dT = f_x dx + f_y dy + f_z dz$
  for $V = g(l, w, h)$, then $dV = g_l dl + g_w dw + g_h dh$.

Example problems:

1. Find the equation of the tangent plane to the surface $z = x^2 - y^2$ at the point $(2, 1)$.
2. Find the formula for the linear approximation of $f(x, y) = x^2 - y^2$ at the point $(2, 1)$.
3. Find the differential of z for $z = x^2 - y^2$. Use this to estimate the change in z if initially, $x = 2$ and $y = 1$, but then x is decreased by 0.05 and y is increased by 0.1.
4. if a cardboard box has dimensions $24in. \times 12in. \times 16in.$ (l × w × h), how does the volume change in we increase the width by 1 inch, decrease the height by 1 inch, and decrease the length by 2 inches? (hint: $V = lwh$ thus $dV = wh\ dl + lh\ dw + lw\ dh$.) How does its surface area change?

12.8 Maximum/minimum problems

Summary of topics and terminology:

- Critical point is where $f_x = 0$ and $f_y = 0$.
- 2^{nd} derivative test: $D = f_{xx} f_{yy} - f_{xy}^2$
  - ⋆ If $D > 0$ and $f_{xx} < 0$ then local max
  - ⋆ If $D > 0$ and $f_{xx} > 0$ then local min
  - ⋆ If $D < 0$ the saddle
  - ⋆ $D = 0$ is inconclusive
- To find absolute min/max (usually over a finite region), first find all local min/max as above, then evaluate the function on the boundary and find all min/max of that.

Example problems:

1. Find all critical points of $f(x, y) = x^2 - y^2$ and classify as to min/max/saddle/inconclusive.
2. Find all critical points of $f(x, y) = 4x^2 + y^2$ and classify as to min/max/saddle/inconclusive.
3. Find all critical points of $f(x, y) = 1 - x^2 - 3y^2$ and classify as to min/max/saddle/inconclusive.
4. Find all critical points of $f(x, y) = 1 - x^2y^2$ and classify as to min/max/saddle/inconclusive.
5. Find absolute max and min of $f(x, y) = x^2 + 1$ over the region $[0, 5] \times [0, 4]$.
6. Find absolute max and min of $f(x, y) = x^2 - y^2$ over the region $[0, 2] \times [0, 1]$.