Chapter 13 – Summary and Review (draft: 2019/03/31-19:50:15)

13.1 Double integrals over rectangles

Summary of topics and terminology:

- Differential of area in the xy-plane: $dA = dxdy = dydx$
- $R = [a,b] \times [c,d]$ then $\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$
- Separable: if $f(x,y) = g(x) \cdot h(y)$ then $\iint_R f(x,y)dA = \int_a^b g(x)dx \cdot \int_c^d h(y)dy$

Example problems:

1. Integrate $f(x,y) = 2x + 3y^2$ over $[0,1] \times [0,5]$.
2. Calculate $\int_0^2 \int_1^3 \ 6xy \ dy \ dx$.

13.2 Double integrals over general regions

Summary of topics and terminology:

- Determine best order of integration, be able to integrate both ways: $dxdy$ and $dydx$.
- Break region up into pieces.
- Be able to solve for intersections of basic curves sich as lines, parabolas, circles, ellipses, etc.
- Sometimes you need to solve to the intersection of curves to get limits of integration.
- $\iint_D 1dA$ = area of region D (numerically, at least, since the double integral result would technically have units of volume still)
- Volume of a solid above region D in xy-plane with "roof" given by $f(x,y)$ is given by $\iint_D f(x,y)dA$

Example problems:

1. Illustrate region of integration D bounded by $y = x^2$ and $y = 2x + 3$. Integrate $f(x,y) = 2y$ over D. (Hint: 1 order of integration is much easier than the other.)
2. Rewrite integral by changing order of integration: $\int_0^1 \int_0^y f(x,y)dxdy$
3. Integrate $f(x,y) = x + y$ over the region bounded by $y = \sqrt{x}$ and $y = x$.
4. Find the volume of the tetrahedron with corners $(0,0,0), (5,0,0), (0,1,0), (0,0,2)$.
5. Find the volume of the region between surfaces given by $f(x,y) = x + y^2$ and $g(x,y) = 1 - x - 2y^2$ that is bounded by the rectangle $[0,1] \times [0,1]$.

13.3 Double integrals in polar coordinates

Summary of topics and terminology:

- Differential of area $dA = r \ dr \ d\theta$
- Even if an integral is given $\iint \ldots \ dx \ dy$, we can realize that $dx \ dy = dA$ and then change it to $\iint \ldots \ rdr \ d\theta$

- Be able to substitute in $x = r\cos\theta$ and $y = r\sin\theta$ into $f(x, y)$ to turn it into polar coordinates.
- Be able to sketch regions of integration that work well with polar coordinates: circles/discs, part of a circle/disc, space between circles, etc.
- Be able to convert an integral from rectangular to polar coordinates and vice versa. This involves being able to decipher the region of integration in one coordinate system (from the limits of integration), and to convert its equations to the other coordinate system.

Example problems:

1. Sketch the region $r \leq 1$, $\pi/2 \leq \theta \leq \pi$.
2. Sketch the region and convert the equations to polar coordinates: $x^2 + y^2 \leq 9$, $y \leq 0$.
3. Integrate $\iint_D e^{-x^2-y^2}\, dA$ where D is the region between circles centered at the origin with radii 1 and 5.
4. Convert to polar coordinates and integrate: $\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} \frac{x^2 y}{x^2+y^2}\, dy\, dx$. (Note that there is seemingly an issue at the origin which gives $\frac{0}{0}$, but there is not actually a problem at all. Just convert to polar coordinates and evaluate.)
5. Find the area enclosed by the spiral $r = \theta$ for $0 \leq \theta \leq \pi$.
6. Find the area of the circle $r = \cos\theta$.

13.4 Triple integrals

Summary of topics and terminology:

- Boxes: $[x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$
- Volume of a region:$Vol(E) = \iiint_E dV$.
- Average value of a function: $\frac{1}{Vol(D)} \iiint_D f(x, y, z)\, dV$.
- Be able to convert between different orders of integration: $dxdydz$, $dydxdz$, $dzdxdy$, etc. There are 6 total.

Example problems:

1. Integrate $\int_0^5 \int_0^2 \int_0^3 2xyz\, dx\, dy\, dz$. (Hint: it's separable.)
2. Write the volume of the tetrahedron with corners $(0, 0, 0), (5, 0, 0), (0, 1, 0), (0, 0, 2)$ as a triple integral. Write it in a few different orders of integration.
3. Convert order of integration to $dx\, dy\, dz$ for the integral $\int_0^{\frac{1}{2}} \int_0^{\frac{1}{3}(1-2x)} \int_0^{1-2x-3y} f(x, y, z)\, dz\, dy\, dx$.
4. Write the integral $\iiint_E f(x, y, z)\, dV$ in all 6 orders of integration where E is the region bounded by the planes $z = 3 - 3y$, $z = 0$, $x = 0$, and parabolic cylinder $4y = x^2$. Here is a 3d plot of the region: http://www.math3d.org/a9dTHO8o. It's not a perfect plot because the bounding surfaces extend beyond the region, but it should still be helpful to you.

13.5 Triple integrals in cylindrical and spherical coordinates

Summary of topics and terminology:

- Cylindrical coords are polar coords plus z
- Cylindrical: $dV = r\ dr\ d\theta\ dz$
- Cylindrical coords often used when 3D region is part of a circular cylinder
- Spherical coords is more like the 3d generalization of polar coords
- Spherical coords: $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi$
  ϕ is angle from positive z-axis,
  θ is as in polar coords, it is the angle from positive x-axis,
  ρ is distance from origin (in 3D). Note that this is slightly different from the polar r.
- Spherical: $dV = \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$
- Spherical coords often used with 3D region of integration is part of a spherical ball.
- Be able to at least roughly sketch in 3D a variety fo regions. This should help with understanding the correct way to write the integral.
- Be able to convert equations from rectangular to cylindrical and spherical coords and back to rectangular.
- Try to become quick at realizing $a \leq x^2 + y^2 \leq b$ is the same as $\sqrt{a} \leq r \leq \sqrt{b}$
  and that $a \leq x^2 + y^2 + z^2 \leq b$ is the same as $\sqrt{a} \leq \rho \leq \sqrt{b}$
- Think about how $z \geq 0$ means that $0 \leq \phi \leq \pi/2$
  and that $z \leq 0$ means that $\pi/2 \leq \phi \leq \pi$
- Consider variations of the above two points, e.g. $\{z \geq 0, y \leq 0\}$ what are the ranges of ϕ and θ?
- Study and recall old trigonometry knowledge. Sine and cosine are periodic, and an integral over a complete period for them cancels out to zero. You may see this in some situations.

Example problems:

1. Sketch the region in 3D given by $E = \{1 \leq x^2 + y^2 \leq 9, 0 \leq z \leq 1\}$. Write equations that give this region in terms of cylindrical coordinates.
2. For the region above, write the integral $\iiint_E y + 2z dV$ in both rectangular and cylindrical coordinates. Evaluate the integral with cylindrical coordinates.
3. Try to sketch in 3D the region $E = \{x^2 + y^2 + z^2 \leq 4, y \geq 0, z \leq 0\}$. Write equations that give this region in terms of spherical coordinates.
4. For the region above, write the integral $\iiint_E x dV$ in both rectangular and spherical coordinates. Evaluate the integral with spherical coordinates.