SPOILER ALERT!!!
DO NOT proceed
until you have attempted to
work the review problems!!!

Chapter 14 §1,2,3 – Summary and Review (draft: 2019/04/15-16:41:15)

14.1 Vector fields

Summary of topics and terminology:

- 2D vector field: $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$
- 3D vector field: $\mathbf{F}(x, y, y) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$
  $= P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$
- Radial vector field: 2D $\mathbf{F} = \langle x, y\rangle = \mathbf{r}$
  3D $\mathbf{F} = \langle x, y, z\rangle = \mathbf{r}$
  both point away form the origin and grow in magnitude as we move away from the origin.
- Other radial vector fields:
  - $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|}$ (constant magnitude)
  - $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^k}$ ($k > 1$ magnitude decays as we move away from origin,
    $k \leq 0$ magnitude grows as we move away from origin.)
- rotational vector fields: Clockwise: $\mathbf{F} = \langle y, -x\rangle$ (negative orientation)
  counter-clockwise: $\mathbf{F} = \langle -y, x\rangle$ (positive orientation)
- Be able to match vector field formulas with their graphs
- Gradient vector field $\mathbf{F} = \nabla f$, f is called a potential function.
- A gradient vector field is conservative.
- Be able to find a potential function or show that a vector field is not a gradient field.
- $\mathbf{F} = \langle P, Q\rangle$ is a gradient field if $P_y = Q_x$. This ties back to Clairaut's theorem that says that $f_{xy} = f_{yx}$.
- To find a potential function, we set $f(x, y) = \int P dx + h(y)$, then take the derivative w.r.t. y of the result of that integral, set that equal to Q,a nd solve for $h(y)$.

Example problems:

1. Find the gradient vector field for $f(x, y) = 3x^2y + xy^3$.

   Solution:

   $\nabla f = \langle f_x, f_y\rangle = \langle 6xy + y^3, 3x^2 + 3xy^2\rangle$.

2. Find the gradient vector field for $f(x, y, z) = xyz$.

   Solution:

   $\nabla f = \langle f_x, f_y, f_z\rangle = \langle yz, xz, xy\rangle$.

3. Determine if the vector field $\mathbf{F} = \langle 2x, 3y^2\rangle$ is a gradient field, and if so, find a potential function f.

   Solution:

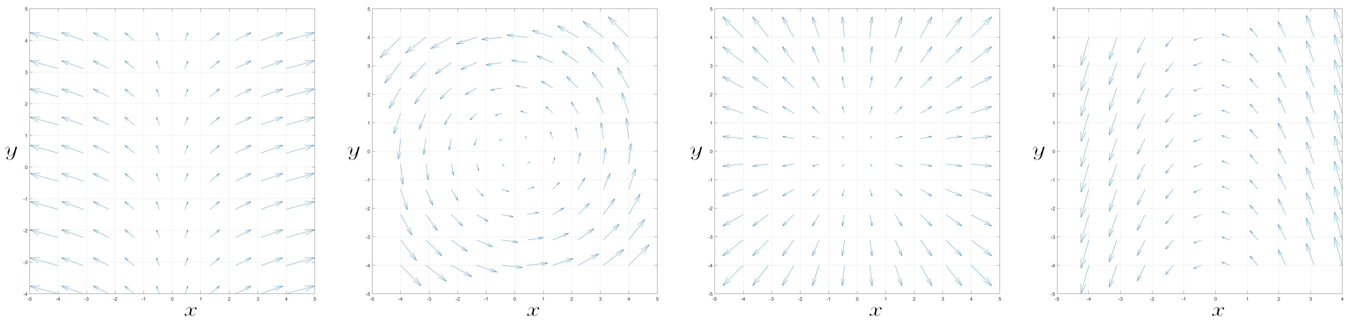
   $f_x = 2x$ thus $f(x, y) = x^2 + h(y)$, so $f_y = h'(y)$ but we want this to equal $3y^2$ thus $h(y) = y^3$. So the potential function is $f(x, y) = x^2 + y^3$.

4. Show that the vector field $\mathbf{F} = \langle xy, -xy^2 \rangle$ is not conservative.

Solution:

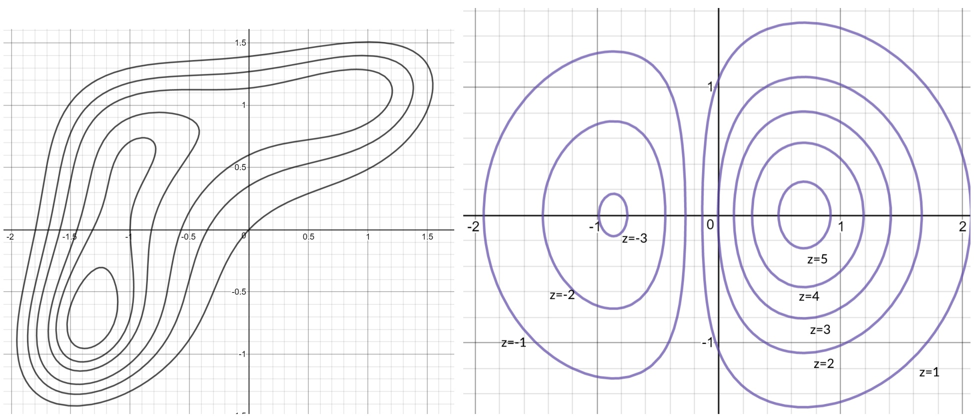
$\frac{\partial}{\partial y} xy = x \neq \frac{\partial}{\partial x}(-xy^2) = -y^2$ thus $P_y \neq Q_x$ so this is not a gradient vector field.

5. Match the vector field formulas with their graphs:
(a) $\mathbf{F} = \langle -1, x \rangle$ (b) $\mathbf{F} = \langle -y, x \rangle$ (c) $\mathbf{F} = \langle x, y \rangle$ (d) $\mathbf{F} = \langle x, 1 \rangle$



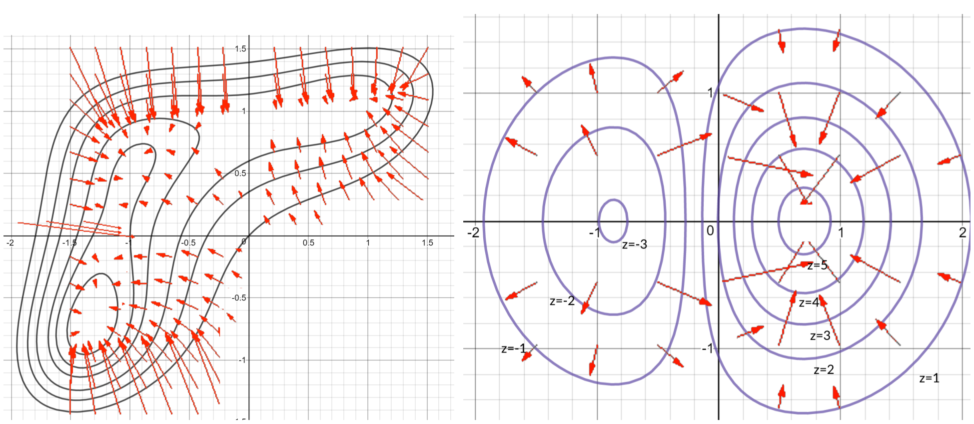
Solution: d, b, c, a

6. Sketch in some vectors for the gradient vector field of two function $z = f(x, y)$ (left) and $z = g(x, y)$ (right) whose contours are plotted below. For f, the outermost contour is the lowest z value. For g, the contours are labeled. Recall that contours close together indicate a steep surface, and contours further apart indicate a more gentle slope.



Solution:

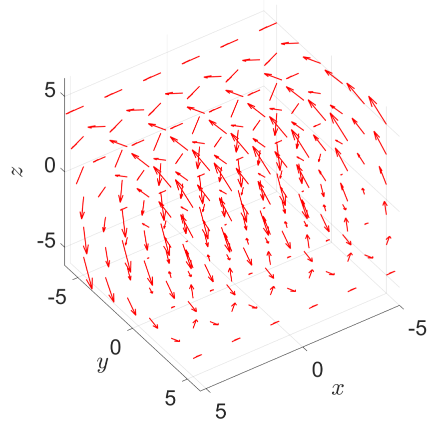
The vectors fir f will all point "inward" in the direction of steepest ascent and perpendicular to the contours. The vectors will be longer where the contours are close together and shorter where the contours are farther apart. The vectors for g will be "outward on the left side (pointing out of a valley), and inward on the right side, pointing towards a peak.



7. Describe the vector field $\mathbf{F} = \langle 1, -z, y \rangle$.

   Solution:

   There is counter-clockwise rotational motion in parallel to the yz plane (counterclockwise relative to looking down from the positive x-axis), and all arrows point slightly in the positive x-direction.



14.2 Line integrals

Summary of topics and terminology:

- Be able to parametrize curves, 2D: $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ and 3D: $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.
- Differential of arc-length: $ds = |\mathbf{r}'(t)|dt$
- $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$ or $\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$
- Plugging in a vector function into a scalar 2D function: $f(\mathbf{r}(t)) = f(x(t), y(t))$. This gives the part of a surface that is above the curve C given by vector function $\mathbf{r}(t)$. I like to think of it being like creating a "fence" above curve C and below the surface.
- In 3D, $f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$.
- Line integral of f over C: $\int_C f ds$.
- $\int_C f ds = \int_a^b f(\mathbf{r}(t)) \, |\mathbf{r}'(t)|dt$.
- Interpretation of line integral of scalar function: I like to think this gives the area of the fence. It can give a negative answer too, just like single integrals in calculus.
- Line integral of a vector field over curve C: $\int_C \mathbf{F} \cdot d\mathbf{r}$.
- Interpretation of line integral of vector field: We can think of it actually as a line integral of a scalar function, where the scalar function is actually the component of the vector field that is tangent to the *oriented* curve. If the vector field general points along the curve, the integral will be positive. If the vector field generally points against the curve, the integral will be negative. You should be sure to understand this interpretation.
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds$ (where $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ is the unit tangent vector)
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy$ with $\mathbf{F} = \langle P, Q \rangle$ and $d\mathbf{r} = \langle dx, dy \rangle$
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)dt$

- $\int_C Pdx + Qdy = \int_{x_0}^{x_1} P(x, f(x))dx + \int_{y_0}^{y_1} Q(f^{-1}(y), y)dy$ where $y = f(x)$ is a way to represent the curve C.

- Be sure to be able to parametrize curves, especially, circles, helices, lines, etc.
  Circle: $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $0 \le t \le 2\pi$.
  Helix: $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ for $0 \le t$. (spirals counter-clockwise rel. to xy-plane and up along z-axis. There are many variations of this.
  Line segment: $\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle (1-t) + \langle x_1, y_1, z_1 \rangle t$ for $0 \le t \le 1$.

- Be able to tell graphically when $\mathbf{F} \cdot \mathbf{T}$ and $\mathbf{F} \cdot \mathbf{n}$ are positive, negative, or zero.

- For $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, we have that $\mathbf{T} = \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} \langle x'(t), y'(t) \rangle$, and
  $$\mathbf{n} = \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} \langle y'(t), -x'(t) \rangle$$

Example problems:

1. Evaluate $\int_C f ds$ where $f(x, y) = x + y^2$ along the line segment connecting $(2, 3)$ to $(5, 1)$.

   Solution:

   The line segment: $\mathbf{r}(t) = \langle 2, 3 \rangle (1-t) + \langle 5, 1 \rangle t = \langle 2+3t, 3-2t \rangle$, for $0 \le t \le 1$ thus $\mathbf{r}' = \langle 3, -2 \rangle$ and $|\mathbf{r}'| = \sqrt{13}$.

   $ds = |\mathbf{r}'| dt = \sqrt{13}\ dt$.

   $f(x, y) = x + y^2$ with $x(t) = 2 + 3t$ and $y(t) = 3 - 2t$ so $f(\mathbf{r}(t)) = 2 + 3t + (3 - 2t)^2$

   So we have that $\int_C f ds = \int_0^1 (2 + 3t + (3 - 2t)^2)\ \sqrt{13}\ dt$
   $$= 2t - \frac{3}{2}t^2 + \frac{1}{3}(3 - 2t)^3 \frac{1}{-2} \Big|_0^1 = 2 - \frac{3}{2} - \frac{1}{6} + \frac{1}{2} = \frac{5}{6}$$

2. Evaluate $\int_C f dx$ for the same function and curve above.

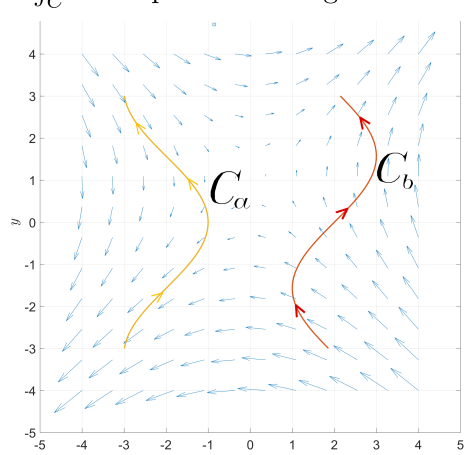
   Solution:

   The line segment $\mathbf{r}(t) = \langle 2 + 3t, 3 - 2t \rangle$ can be written as $y = \frac{-2}{3}(x - 2) + 3$ with $2 \le x \le 5$.

   So we integrate $\int_C f dx = \int_2^5 f\left(x, \frac{-2}{3}(x - 2) + 3\right) dx = \int_2^5 \left[x + \left(\frac{-2}{3}(x - 2) + 3\right)^2\right]\ dx$,

   $\int_2^5 \left[x + \left(\frac{-2}{3}(x - 2) + 3\right)^2\right]\ dx = \int_2^5 (x + \frac{1}{9}(13 - 2x)^2)\ dx = \frac{1}{2}x^2 - \frac{1}{54}(13 - 2x)^3|_2^5 = \frac{47}{2}$

3. Is $\int_C \mathbf{F} \cdot d\mathbf{r}$ positive or negative?



   Solution:

   $\int_{C_a} \mathbf{F} \cdot d\mathbf{r} < 0$ for curve a since the vector field generally points in the 'backward direction' along the curve. Generally the angle between the vector field vectors and the tangent vector

tho the curve will be between $\pi/2$ and π.
$\int_{C_b} \mathbf{F} \cdot d\mathbf{r} > 0$ for curve b since the vector field generally points in the 'forward direction' along the curve. Generally the angle between the vector field vectors and the tangent vector to the curve will be between 0 and $\pi/2$.

4. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle 2x, 3y+1 \rangle$ and $\mathbf{r} = \langle t, t^2 \rangle$ and $0 \le t \le 1$.

   Solution:

   $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 2t, 3t^2+1 \rangle \cdot \langle 1, 2t \rangle \; dt = \int_0^1 2t + 6t^3 + 2t \; dt = 2t^2 + \frac{6}{4}t^4\Big|_0^1 = 3.5$

   This is a positive value, so if we were to plot this vector field and curve,we would see that the vectors generally point in the forward direction along the curve.

5. Calculate the circulation of $\mathbf{F} = \langle y, x \rangle$ on the unit circle (oriented positively).

   Solution:

   The unit circle is parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le 2\pi$.

   This is a nice curve since $|\mathbf{r}'(t)| = 1$, thus $ds = |\mathbf{r}'(t)|dt = 1 \; dt$.

   $$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^{2\pi} \langle \sin(t), \cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \\ &= \int_0^{2\pi} (-\sin^2(t) + \cos^2(t)) dt = \int_0^{2\pi} \cos(2t) dt = 0\end{aligned}$$

   Note that we have used the double-angle trig identities above.

   If you graph this vector field, you'll see that it flows with the curve on the left and right sides, but against the curve on the top and bottom. Thus it makes sense that the net circulation cancels out.

6. Calculate the (outward) flux of $\mathbf{F} = \langle y, x \rangle$ across the unit circle (oriented positively).

   Solution:

   As above, the unit circle is parametrized by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le 2\pi$. But we need to find the outward normal vector. So we add a zero $\mathbf{k}$ component and cross with the $\mathbf{k}$ unit vector:

   $\mathbf{n} = \mathbf{T} \times \mathbf{k} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \times \mathbf{k} = \langle -\sin t, \cos t, 0 \rangle \times \langle 0, 0, 1 \rangle = \langle \cos t, \sin t, 0 \rangle$

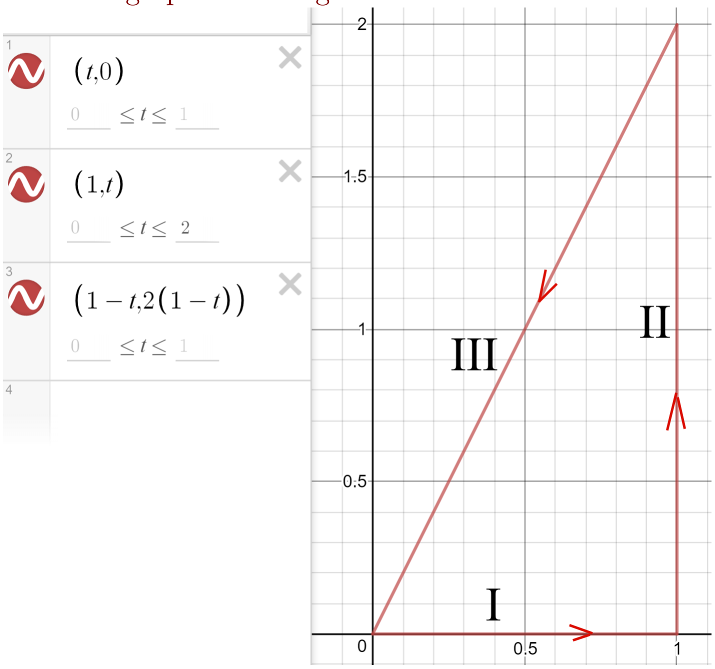
   $$\begin{aligned}\text{Thus } \int_C \mathbf{F} \cdot \mathbf{n} \; ds &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \langle \cos t, \sin t, 0 \rangle \; 1 \; dt \\ &= \int_0^{2\pi} \langle \sin t, \cos t, 0 \rangle \cdot \langle \cos t, \sin t, 0 \rangle \; dt \\ &= \int_0^{2\pi} 2 \sin(t) \cos(t) dt \\ &= \sin^2(t)|_0^{2\pi} = 0\end{aligned}$$

   So in addition to having zero net circulation around the unit circle, this field also has zero flux across it! Again, plotting the vector field provides some insight. There will be inward flux from the NW and SE corners and outward flux in the NE and SW corners that cancel each other out.

7. Calculate the flux and circulation of $\mathbf{F} = \langle -xy, 1 \rangle$ for the triangular loop $(0,0), (1,0), (1,2)$.

   Solution:

Here is a graph of the region from Desmos:



Curve sections and parametrizations:
I) $\mathbf{r}(t) = \langle t, 0\rangle$ for $0 \le t \le 1$. $\mathbf{T} = \langle 1, 0\rangle$ and $\mathbf{n} = \langle 0, -1\rangle$
II) $\mathbf{r}(t) = \langle 1, t\rangle$ for $0 \le t \le 2$. $\mathbf{T} = \langle 0, 1\rangle$ and $\mathbf{n} = \langle 1, 0\rangle$
III) $\mathbf{r}(t) = \langle 1-t, 2(1-t)\rangle$ for $0 \le t \le 1$. $\mathbf{T} = \frac{1}{\sqrt{5}}\langle -1, -2\rangle$ and $\mathbf{n} = \frac{1}{\sqrt{5}}\langle -2, 1\rangle$

Circulation:

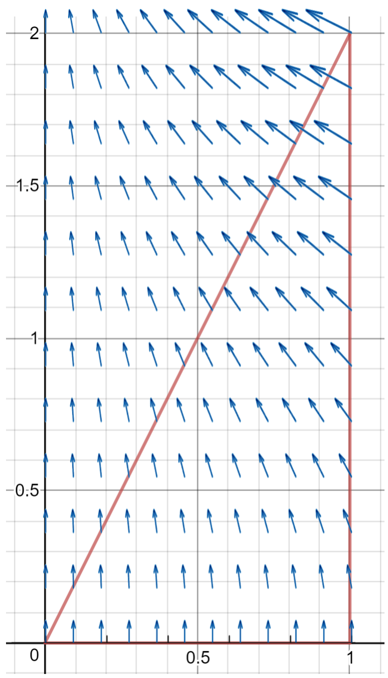
$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{C_I} F(\mathbf{r}(t))\cdot \mathbf{r}'(t)\ dt + \int_{C_{II}} F(\mathbf{r}(t))\cdot \mathbf{r}'(t)\ dt + \int_{C_{III}} F(\mathbf{r}(t))\cdot \mathbf{r}'(t)\ dt \\ &= \int_0^1 \langle 0, 1\rangle \cdot \langle 1, 0\rangle\ dt + \int_0^2 \langle -t, 1\rangle \cdot \langle 0, 1\rangle\ dt + \int_0^1 \langle -2(1-t)^2, 1\rangle \cdot \langle -1, -2\rangle\ dt \\ &= \int_0^1 0\ dt + \int_0^2 1\ dt + \int_0^1 2(1-t)^2 - 2\ dt \\ &= 0 + 2 + \left[-\frac{2}{3}(1-t)^3 - 2t\right]_0^1 = 2 - 2 + \frac{2}{3} = \frac{2}{3}\end{aligned}$$

Flux:

$$\begin{aligned}\int_C \mathbf{F}\cdot \mathbf{n}\ ds &= \int_{C_I} F(\mathbf{r}(t))\cdot \mathbf{n}\ |\mathbf{r}'(t)|\ dt + \int_{C_{II}} F(\mathbf{r}(t))\cdot \mathbf{n}\ |\mathbf{r}'(t)|\ dt + \int_{C_{III}} F(\mathbf{r}(t))\cdot \mathbf{n}\ |\mathbf{r}'(t)|\ dt \\ &= \int_0^1 \langle 0, 1\rangle \cdot \langle 0, -1\rangle\ 1\ dt + \int_0^2 \langle -t, 1\rangle \cdot \langle 1, 0\rangle\ 1\ dt + \int_0^1 \langle -2(1-t)^2, 1\rangle \cdot \frac{1}{\sqrt{5}}\langle -2, 1\rangle\ \sqrt{5}\ dt \\ &= \int_0^1 (-1)\ dt + \int_0^2 (-t)\ dt + \int_0^1 4(1-t)^2 + 1\ dt \\ &= -1 - 2 + \left[-\frac{4}{3}(1-t)^3 - t\right]_0^1 = -3 - 1 + \frac{4}{3} = -\frac{13}{3}\end{aligned}$$

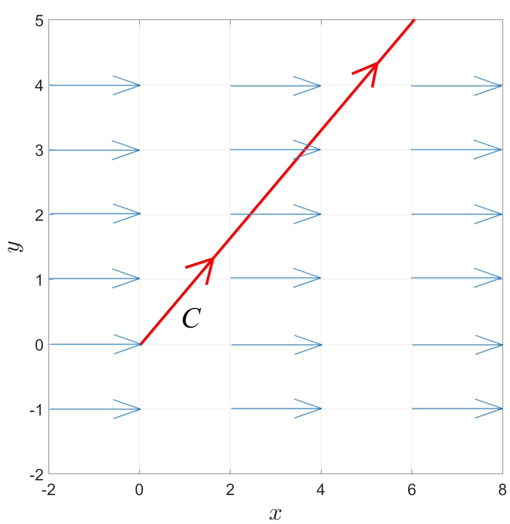
So the vector field $\mathbf{F}$ overall has an inward flux into the triangle, but it has a general positive flow around the boundary of the triangular region. This makes intuitive sense because x and y are both positive, so $-xy$ is negative. Thus $\mathbf{F}$ has a negative horizontal component, so the vectors point from the SE to the NW generally. Furthermore the vector going into the right side of the triangle are longer than those exiting along the hypotenuse. And vectors are entering the triangle along the sides I and II, and only leaving the triangle along side III, thus overall there is an inward flow/flux.

Here is a graph with the vector field overlaid on to the triangle:



Looking at the above graph, we can see that the vectors are perpendicular tot he bottom of the triangle, so there is 0 circulation there, and there is positive circulation along the right side since the vectors generally point upward there. Along the hypotenuse of the triangle, the vectors generally are almost perpendicular or point against the flow downward along the curve, thus we get a negative component to the circulation there, but these vectors are shorter than those on the right side. So overall, we have a positive circulation along the triangle.

8. Estimate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ from the graph. Assume that everything is plotted exactly to scale.



Solution:

The curve is a line $y = \frac{5}{6}x$, and the vector field is the same everywhere. The vectors all point right with length 2.

$\int_C \mathbf{F} \cdot d\mathbf{r} =$ integral of the projection of $\mathbf{F}$ onto C

So: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C |\mathbf{F}| \cos(\theta) ds = \int_C 2\cos(\theta) ds$

We just need to figure out what the cosine of the angle between the line and vector field is, but this is just the horizontal run of the line divided by its length: $\cos\theta = \frac{6}{\sqrt{61}}$.

Thus $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C |\mathbf{F}| \cos(\theta) ds = \int_C 2 \cdot \frac{6}{\sqrt{61}} ds = \frac{12}{\sqrt{61}} \cdot (\text{length of } C) = \frac{12}{\sqrt{61}} \cdot \sqrt{5^2 + 6^2} = 12$.

We can confirm this be seeing that $\mathbf{F} = \langle 2, 0\rangle$ and $\mathbf{r}(t) = \langle 6t, 5t\rangle$ for $0 \le t \le 1$.

14.3 Conservative vector fields

Summary of topics and terminology:

- $\mathbf{F} = \langle P, Q\rangle$ is conservative if and only if $P_y = Q_x$.
- $\mathbf{F} = \langle P, Q, R\rangle$ is conservative if and only if $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$.
- $\mathbf{F}$ is conservative means that there is a function f such that $\mathbf{F} = \nabla f$
- $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ if $\mathbf{F}$ is conservative and C is a closed curve.
- Fundamental theorem of line integrals: $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ where C is any curve that goes from $\mathbf{r}(a)$ to $\mathbf{r}(b)$.
- Be able to break loops up into different pieces and parametrize each.
- $\mathbf{F}$ is path-independent if the value of the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ does not depend on the shape of the curve, but only on the starting and ending points.
- A vector field is path-independent if and only if it is conservative.
- Understand that a closed curve can be broken into different pieces: $C = C_1 \cup C_2$ and that the integral over C is the sum of the integrals over C_1 and C_2. Of course we need to be careful to keep the orientations. $\int_{C_1 \cup C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.
- If we reverse the orientation of a curve C (this can be denoted by $-C$), the the integral flips sign: $\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$.

Example problems:

1. Show that $\mathbf{F} = \langle y^2, 2xy - 1\rangle$ is conservative and find potential function f.

   Solution:

   If $\mathbf{F} = \nabla f$, then $\mathbf{F} = \langle P, Q\rangle$ and $P_y = Q_x$.

   $P_y = \frac{\partial}{\partial y} y^2 = 2y$ and $Q_x = \frac{\partial}{\partial x}(2xy - 1) = 2y$ so it is indeed a conservative vector field.

   We need to find a potential f such that $\mathbf{F} = \langle f_x, f_y\rangle$ with $f_x = y^2$ and $f_y = 2xy - 1$.

   We integrate f_x w.r.t. x: $\int y^2 \, dx = xy^2 + h(y)$. We need to add an arbitrary funciton of y as our "constant" of integration here.

   So $f(x, y) = xy^2 + h(y)$. Now differentiate w.r.t. y and set it equation to the 2nd component of our vector field. $f_y = 2xy + h'(y)$ thus we must have that $2xy + h'(y) = 2xy - 1$ so $h'(y) = -1$ and $h(y) = -y$.

   Now we have that $f(x, y) = xy^2 - y$ and $\mathbf{F} = \nabla f$.

2. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F}$ above and C is a curve that goes form $(0,1)$ to $(3,2)$.

   Solution:

   We have that $f(x,y) = xy^2 - y$ and $\mathbf{F} = \nabla f$. By the fundamental theorem of line integrals:

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

   so

$$\int_C \nabla f \cdot d\mathbf{r} = f(3,2) - f(0,1) = (3 \cdot 2^2 - 2) - (0 \cdot 1^2 - 1) = 12 - 2 - 0 + 1 = 11$$

3. With the same vector field $\mathbf{F}$ as above, calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the triangle with vertices $(0,0), (0,3), (3,5)$.

   Solution:

   $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ because $\mathbf{F}$ is conservative and C is a closed loop.

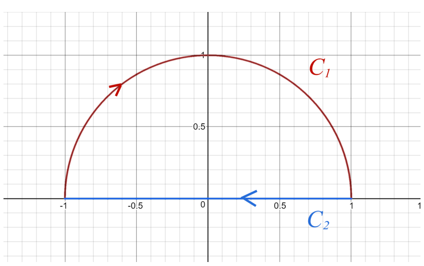
4. Assume that C_1 is the upper semi-circle that goes from $(-1,0)$ to $(1,0)$, and $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 5$, and $\mathbf{F}$ is path-independent. Calculate $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ where C_2 is the line segment that goes from $(1,0)$ to $(-1,0)$.

   Solution:

   We have that $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = -5$ because C_1 and C_2 together form a closed loop and thus $0 = \int_{C_1 \cup C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

   Since $\mathbf{F}$ is path-independent, we also know that it is conservative and is thus a gradient vector field.

   See the figure below.



5. If $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$, for vector field $\mathbf{F}$ and closed curve C, are we guaranteed that $\mathbf{F}$ is conservative?

   Solution:

   No! We are only guaranteed that $\mathbf{F}$ is conservative (path-independent) if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for **ALL** closed curves C.

6. Consider the curves: C_1 is the line segment from $(0,0)$ to $(1,1)$, C_2 is the line segment from $(1,1)$ to $(-5,1)$, and C_3 is the line segment from $(-5,1)$ to $(0,0)$. If we know that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 2$, $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -3$, and $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 8$, then calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the triangle with vertices $(0,0), (1,1), (-5,1)$ oriented clockwise.

   Solution:

   Note that the orientation is flipped for the triangle C as opposed to the original line segments mentioned in the statement of the problem.

   We have that $C_1 \cup C_2 \cup C_3$ make up the triangle we are interested in, but collectively, our line segments create a positively-oriented triangle (counter-clockwise). We are supposed to calculate the integral over the triangle in clockwise fashion. So we will need to add the integrals of our line segments and then multiple that by a negative.

$\int_C \mathbf{F} \cdot d\mathbf{r} = -(2 - 3 + 8) = -7$

Also note that this makes no assumption about the vector field. In fact, since we have a closed loop where the integral is non-zero, we are guaranteed that this vector field is not conservative.