Chapter 14 §1,2,3 – Summary and Review (draft: 2019/04/15-16:42:05)

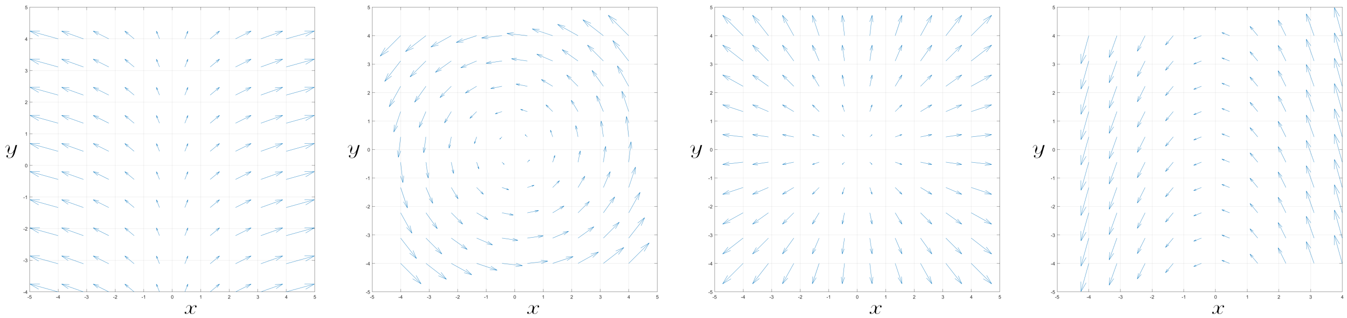
14.1 Vector fields

Summary of topics and terminology:

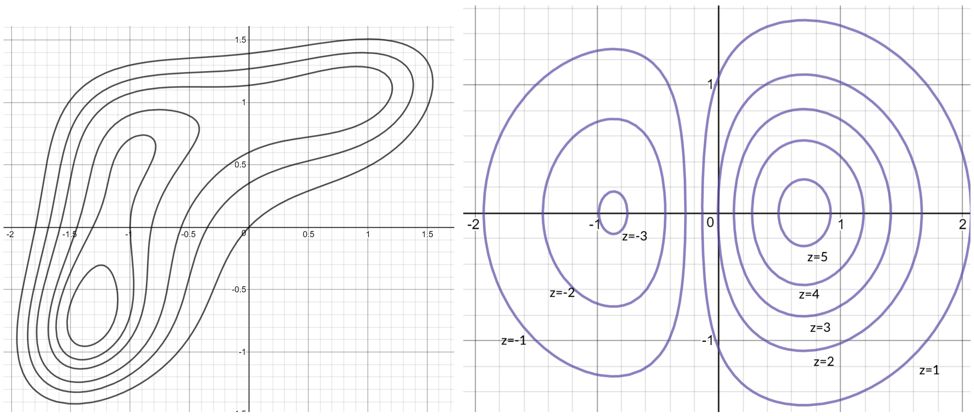
- 2D vector field: $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$
- 3D vector field: $\mathbf{F}(x, y, y) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$
  $= P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$
- Radial vector field: 2D $\mathbf{F} = \langle x, y \rangle = \mathbf{r}$
  3D $\mathbf{F} = \langle x, y, z \rangle = \mathbf{r}$
  both point away form the origin and grow in magnitude as we move away from the origin.
- Other radial vector fields:
  - $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|}$ (constant magnitude)
  - $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^k}$ ($k > 1$ magnitude decays as we move away from origin,
    $k \leq 0$ magnitude grows as we move away from origin.)
- rotational vector fields: Clockwise: $\mathbf{F} = \langle y, -x \rangle$ (negative orientation)
  counter-clockwise: $\mathbf{F} = \langle -y, x \rangle$ (positive orientation)
- Be able to match vector field formulas with their graphs
- Gradient vector field $\mathbf{F} = \nabla f$, f is called a potential function.
- A gradient vector field is conservative.
- Be able to find a potential function or show that a vector field is not a gradient field.
- $\mathbf{F} = \langle P, Q \rangle$ is a gradient field if $P_y = Q_x$. This ties back to Clairaut's theorem that says that $f_{xy} = f_{yx}$.
- To find a potential function, we set $f(x, y) = \int P dx + h(y)$, then take the derivative w.r.t. y of the result of that integral, set that equal to Q,a nd solve for $h(y)$.

Example problems:

1. Find the gradient vector field for $f(x, y) = 3x^2y + xy^3$.
2. Find the gradient vector field for $f(x, y, z) = xyz$.
3. Determine if the vector field $\mathbf{F} = \langle 2x, 3y^2 \rangle$ is a gradient field, and if so, find a potential function f.
4. Show that the vector field $\mathbf{F} = \langle xy, -xy^2 \rangle$ is not conservative.
5. Match the vector field formulas with their graphs:
   (a) $\mathbf{F} = \langle -1, x \rangle$ (b) $\mathbf{F} = \langle -y, x \rangle$ (c) $\mathbf{F} = \langle x, y \rangle$ (d) $\mathbf{F} = \langle x, 1 \rangle$



6. Sketch in some vectors for the gradient vector field of two function $z = f(x, y)$ (left) and $z = g(x, y)$ (right) whose contours are plotted below. For f, the outermost contour is the lowest z value. For g, the contours are labeled. Recall that contours close together indicate a steep surface, and contours further apart indicate a more gentle slope.



7. Describe the vector field $\mathbf{F} = \langle 1, -z, y\rangle$.

14.2 Line integrals

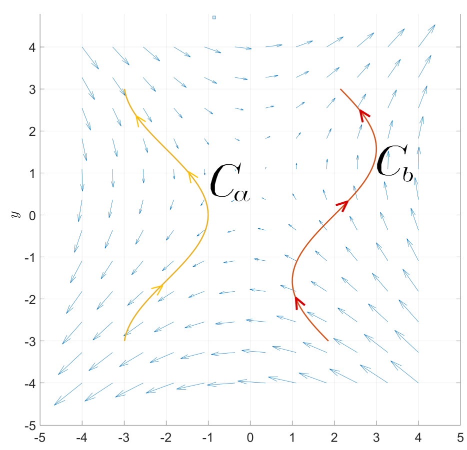
Summary of topics and terminology:

- Be able to parametrize curves, 2D: $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ and 3D: $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$.
- Differential of arc-length: $ds = |\mathbf{r}'(t)|dt$
- $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2}$ or $\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$
- Plugging in a vector function into a scalar 2D function: $f(\mathbf{r}(t)) = f(x(t), y(t))$. This gives the part of a surface that is above the curve C given by vector function $\mathbf{r}(t)$. I like to think of it being like creating a "fence" above curve C and below the surface.
- In 3D, $f(\mathbf{r}(t)) = f(x(t), y(t), z(t))$.
- Line integral of f over C: $\int_C f ds$.
- $\int_C f ds = \int_a^b f(\mathbf{r}(t))\ |\mathbf{r}'(t)|dt$.
- Interpretation of line integral of scalar function: I like to think this gives the area of the fence. It can give a negative answer too, just like single integrals in calculus.
- Line integral of a vector field over curve C: $\int_C \mathbf{F} \cdot d\mathbf{r}$.

- Interpretation of line integral of vector field: We can think of it actually as a line integral of a scalar function, where the scalar function is actually the component of the vector field that is tangent to the *oriented* curve. If the vector field general points along the curve, the integral will be positive. If the vector field generally points against the curve, the integral will be negative. You should be sure to understand this interpretation.
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$ (where $\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ is the unit tangent vector)
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy$ with $\mathbf{F} = \langle P, Q \rangle$ and $d\mathbf{r} = \langle dx, dy \rangle$
- $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t) dt$
- $\int_C P dx + Q dy = \int_{x_0}^{x_1} P(x, f(x)) dx + \int_{y_0}^{y_1} Q(f^{-1}(y), y) dy$ where $y = f(x)$ is a way to represent the curve C.
- Be sure to be able to parametrize curves, especially, circles, helices, lines, etc.
  Circle: $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$ for $0 \le t \le 2\pi$.
  Helix: $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ for $0 \le t$. (spirals counter-clockwise rel. to xy-plane and up along z-axis. There are many variations of this.
  Line segment: $\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle (1 - t) + \langle x_1, y_1, z_1 \rangle t$ for $0 \le t \le 1$.
- Be able to tell graphically when $\mathbf{F} \cdot \mathbf{T}$ and $\mathbf{F} \cdot \mathbf{n}$ are positive, negative, or zero.
- For $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, we have that $\mathbf{T} = \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} \langle x'(t), y'(t) \rangle$, and
  $\mathbf{n} = \frac{1}{\sqrt{(x'(t))^2 + (y'(t))^2}} \langle y'(t), -x'(t) \rangle$

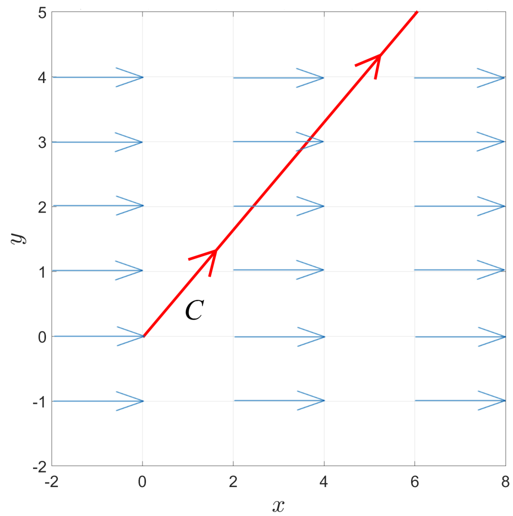
Example problems:

1. Evaluate $\int_C f ds$ where $f(x, y) = x + y^2$ along the line segment connecting $(2, 3)$ to $(5, 1)$.
2. Evaluate $\int_C f dx$ for the same function and curve above.
3. Is $\int_C \mathbf{F} \cdot d\mathbf{r}$ positive or negative?



4. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle 2x, 3y + 1 \rangle$ and $\mathbf{r} = \langle t, t^2 \rangle$ and $0 \le t \le 1$.
5. Calculate the circulation of $\mathbf{F} = \langle y, x \rangle$ on the unit circle (oriented positively).
6. Calculate the (outward) flux of $\mathbf{F} = \langle y, x \rangle$ across the unit circle (oriented positively).
7. Calculate the flux and circulation of $\mathbf{F} = \langle -xy, 1 \rangle$ for the triangular loop $(0, 0), (1, 0), (1, 2)$.

8. Estimate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ from the graph. Assume that everything is plotted exactly to scale.



14.3 Conservative vector fields

Summary of topics and terminology:

- $\mathbf{F} = \langle P, Q \rangle$ is conservative if and only if $P_y = Q_x$.
- $\mathbf{F} = \langle P, Q, R \rangle$ is conservative if and only if $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$.
- $\mathbf{F}$ is conservative means that there is a function f such that $\mathbf{F} = \nabla f$
- $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ if $\mathbf{F}$ is conservative and C is a closed curve.
- Fundamental theorem of line integrals: $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ where C is any curve that goes from $\mathbf{r}(a)$ to $\mathbf{r}(b)$.
- Be able to break loops up into different pieces and parametrize each.
- $\mathbf{F}$ is path-independent if the value of the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ does not depend on the shape of the curve, but only on the starting and ending points.
- A vector field is path-independent if and only if it is conservative.
- Understand that a closed curve can be broken into different pieces: $C = C_1 \cup C_2$ and that the integral over C is the sum of the integrals over C_1 and C_2. Of course we need to be careful to keep the orientations. $\int_{C_1 \cup C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.
- If we reverse the orientation of a curve C (this can be denoted by $-C$), the the integral flips sign: $\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$.

Example problems:

1. Show that $\mathbf{F} = \langle y^2, 2xy - 1 \rangle$ is conservative and find potential function f.
2. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F}$ above and C is a curve that goes form $(0, 1)$ to $(3, 2)$.
3. With the same vector field $\mathbf{F}$ as above, calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the triangle with vertices $(0, 0), (0, 3), (3, 5)$.

4. Assume that C_1 is the upper semi-circle that goes from $(-1, 0)$ to $(1, 0)$, and $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 5$, and $\mathbf{F}$ is path-independent. Calculate $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ where C_2 is the line segment that goes from $(1, 0)$ to $(-1, 0)$.

5. If $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$, for vector field $\mathbf{F}$ and closed curve C, are we guaranteed that $\mathbf{F}$ is conservative?

6. Consider the curves: C_1 is the line segment from $(0, 0)$ to $(1, 1)$, C_2 is the line segment from $(1, 1)$ to $(-5, 1)$, and C_3 is the line segment from $(-5, 1)$ to $(0, 0)$. If we know that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 2$, $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = -3$, and $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 8$, then calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the triangle with vertices $(0, 0), (1, 1), (-5, 1)$ oriented clockwise.