Chapter 14 §4,5 – Summary and Review (draft: 2019/04/22-11:19:38)

14.4 Green's theorem

Summary of topics and terminology:

- All of the following conditions are equivalent:
  - ➠ $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for any closed loops C
  - ➠ $\mathbf{F}$ is path-independent
  - ➠ $\int_C \mathbf{F}\cdot d\mathbf{r}$ is the same value for all paths C that go from point A to point B
  - ➠ $\mathbf{F}$ is conservative
  - ➠ there exists a potential function f such that $\mathbf{F} = \nabla f$.
  - ➠ $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$ where curve C is parameterized by $\mathbf{r}(t)$ for $a \le t \le b$.
- Green's Theorem for vector field $\mathbf{F} = \langle P, Q\rangle$ and simply connected region D with counter-clockwise oriented boundary curve $C = \partial D$.
  - ➠ Circulation form:
$$\oint_{\partial D} \mathbf{F}\cdot d\mathbf{r} = \iint_D (Q_x - P_y)\, dA$$
  - ➠ Flux form:
$$\oint_{\partial D} \mathbf{F}\cdot \mathbf{n}\, ds = \iint_D (P_x + Q_y)\, dA$$
- Use Green's theorem to calculate area:
$$A(D) = \iint_D dA = \oint_{\partial D} x\, dy = -\oint_{\partial D} y\, dx = \frac{1}{2}\oint_{\partial D} x\, dy - y\, dx$$

Example problems:

1. Calculate the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where C is the rectangle bounding region $[0,2]\times[0,3]$ and $\mathbf{F} = \langle 3x^2y, x + y^3\rangle$.

   Solution:

   By Green's theorem: $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_C P dx + Q dy = \iint_D Q_x - P_y\, dA$ where D is the rectangle $[0,2]\times[0,3]$.

   $Q_x - P_y = 1 - 3x^2$ so $\iint_D Q_x - P_y\, dA = \int_0^3\int_0^2 (1-3x^2)\, dx\, dy = 3\cdot(x - x^3)|_0^2 = -18$

   Thus we can think of this as meaning there is a net clockwise flow along the boundary and a net clockwise rotation of the vector field inside the rectangle. If we dropped billions of tiny little paddle wheels, the majority of them would rotate clockwise.

2. Calculate the area of the unit circle using Green's theorem.

   Solution:

   Here are three possible options:

   $A(D) = \oint_{\partial D} x\, dy$

   $A(D) = -\oint_{\partial D} y\, dx$

   $A(D) = \frac{1}{2}\oint_{\partial D} x\, dy - y\, dx$

I'll use the first option:

The unit circle is parameterized: $\mathbf{r}(t) = \langle \cos(t), \sin(t) \rangle$

$\oint_{\partial D} x\ dy = \int_a^b x(t)\ y'(t)\ dt$

$$A(D) = \oint_{\partial D} x\ dy = \int_0^{2\pi} \cos(t)\cos(t)\ dt = \pi$$

3. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle 2xy^2, 2x^2y - 1 \rangle$ and C is given by $\mathbf{r}(t) = \langle \cos(2t) + t^3 \sin(t), \cos(t) - t^2 \sin(t) \rangle$ for $0 \le t \le \pi$. *(hint: Is* $\mathbf{F}$ *path-independent?)*

   <u>*Solution:*</u>

   $P_y = Q_x = 4xy$ so $\mathbf{F}$ is conservative, hence path-independent. So we just need to find a potential function and evaluate it at the endpoints of the curve.

   $\mathbf{F} = \nabla f$ for $f(x, y) = x^2y^2 - y$.

   $\mathbf{r}(0) = \langle 1, 1 \rangle$ and $\mathbf{r}(\pi) = \langle 1, -1 \rangle$

   $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \nabla f \cdot \mathbf{r}'(t)\ dt = f(1, -1) - f(1, 1) = 2 - 0 = 2$

4. Calculate $\displaystyle\oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\ ds$ using Green's theorem for $\mathbf{F} = \langle 2xy, x - y \rangle$ and D is the region bounded by the x-axis and the parabola $y = 1 - x^2$.

   <u>*Solution:*</u>

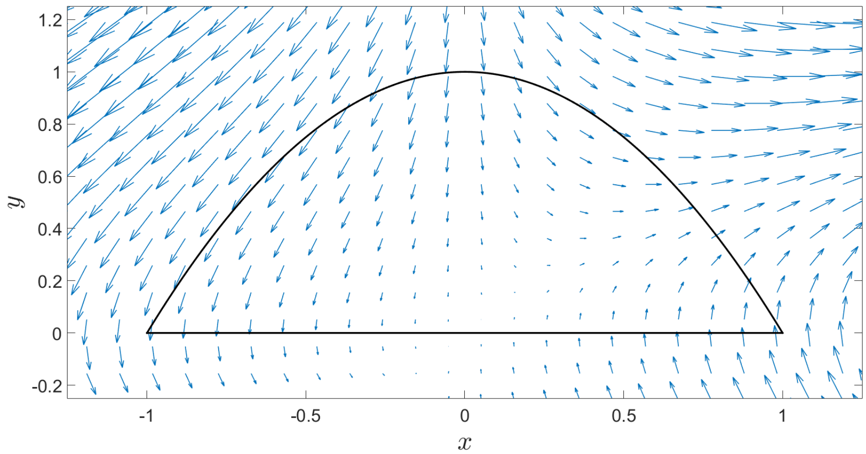
   This uses the flux form of Green's:
   $$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\ ds = \iint_D (P_x + Q_y)\ dA$$

   $$\begin{aligned}\iint_D (P_x + Q_y)\ dA &= \int_{-1}^1 \int_0^{1-x^2} (2y - 1)\ dy\ dx \\ &= \int_{-1}^1 (1 - x^2)^2 - (1 - x^2)\ dx \\ &= \int_{-1}^1 (1 - x^2)(1 - x^2 - 1)\ dx \\ &= \int_{-1}^1 (1 - x^2)(-x^2)\ dx \\ &= \int_{-1}^1 (-x^2 + x^4)\ dx \\ &= 2\int_0^1 (-x^2 + x^4)\ dx \\ &= -\tfrac{2}{3}x^3 + \tfrac{2}{5}x^5 |0^1 = \tfrac{2}{5} - \tfrac{2}{3} = -\tfrac{4}{15}.\end{aligned}$$

   So there is a net negative flux across the boundary and thus the region D is a sink overall.

   Note that not all points in D are necessarily a sink. The divergence of $\mathbf{F}$ is $2y - 1$. This is positive for any point with $y > \frac{1}{2}$. So the upper part of the region consists of sources and the lower part sinks. It just so happens that the sinks are stronger.

   Here is a plot of the vector field and region:



From the graph it is clear that vectors are generally flowing into the region.

Notice that it is really difficult to tell that points with $y > \frac{1}{2}$ are sources, and that being a source or sink is not necessarily correlated with the flux crossing the boundary in a particular direction. This particular vector field has sink behavior vertically since $Q_y = -1 < 0$ but the horizontal source/sink behavior is $P_x = 2y$ which depends on the value of y.

Let's look at the point $(0, 1)$. We have that $\mathbf{F}(0, 1) = \langle 0, -1 \rangle$, so the vector points downward into the region D. If we look slightly to the right, we get $\mathbf{F}(0+\Delta x, 1) = \langle 2\Delta x, \Delta x - 1 \rangle$. Notice that the x component has increased! This is source behavior int eh horizontal direction. Even if we look to the left instead, we'll see that the horizontal component becomes negative. Dropping a floating point near $(0, 1)$ it would get pushed away horizontally, since $(0, 1)$ is a horizontal source.

Now let's look slightly above and below. $\mathbf{F}(0, 1+\Delta y) = \langle 0, -1-\Delta y \rangle$ thus the y component has decreased indicating sink behavior vertically. This is a little easier to assess visually since the vectors point downward and generally get shorter as we follow their flow near $(0, 1)$.

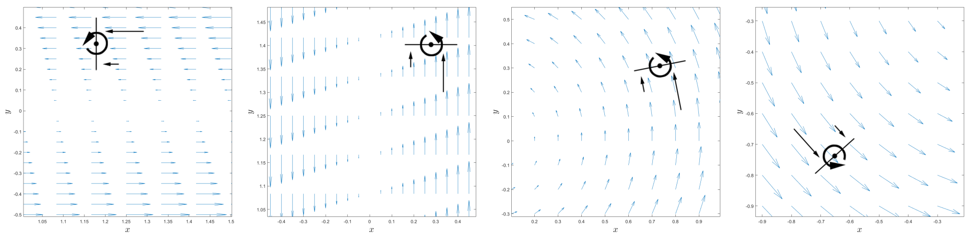
14.5 Divergence and curl

Summary of topics and terminology:

- $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ is a 3D differential operator.
- div $\mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z$. This is a dot product, so divergence is a scalar.
- curl $\mathbf{F} = \nabla \times \mathbf{F} = \langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle$. This is a cross product, and curl is a vector.
- 2D divergence: div $\mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y$
  This is what is inside the double integral in the flux form of Green's theorem.
- 2D curl: curl $\mathbf{F} = Q_x - P_y$. (*this is only in 2D!!)
  Technically the curl is: curl $\mathbf{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0 \rangle \times \langle P, Q, 0 \rangle = (Q_x - P_y)\mathbf{k}$
  Curl is always a vector, generally, but for a vector field in the xy-plane, it will always point along the z-axis so we tend to just take the scalar component.
  This is inside the circulation form of Green's theorem double integral (without the $\mathbf{k}$)
- Curl describes the rotational or twist-like character of a vector field.
  Imagine putting a little paddle wheel in the vector field (and having its axis fixed at that

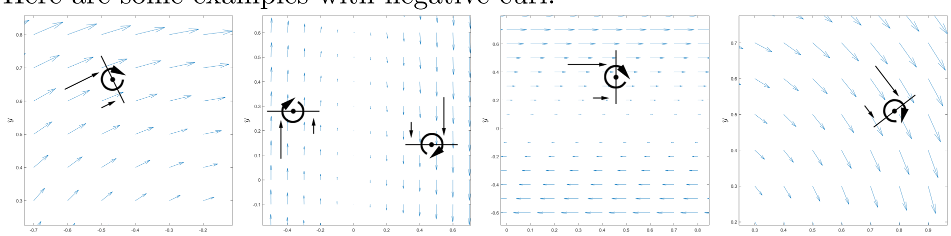
location) and thinking of the vectors as fluid flow velocities. If the force on one side of the paddle wheel is stronger, it will rotate.
It is tricky to really understand what this means, but generally if we move in the positive y-direction, the horizontal component of the vectors decreases, and if we move in the positive x-direction, the vertical component of the vectors increases.

- Here are some examples with positive curl:



From left to right:
(1) $\mathbf{F} = \langle -y, 0 \rangle$. For positive y, the vectors are pointing left, and as we move up, the vectors get longer thus $P_y < 0$. For negative y, the vectors are pointing right, and as we move up, the vectors get shorter thus $P_y < 0$.
(2) $\mathbf{F} = \langle 0, x \rangle$. For positive x, the vectors are pointing up, and as we move right, they get longer thus $Q_x > 0$. For negative x, the vectors are pointing down, and as we move right, they get longer thus $Q_x > 0$.
(3) and (4) $\mathbf{F} = \langle -y, x \rangle$. As we follow the vectors forward, there is a counter-clockwise rotation.

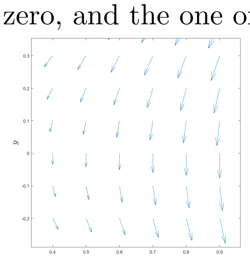
- Here are some examples with negative curl:



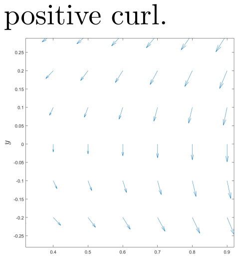
There are very similar to those above, but with the arrows reversed. Generally, there is a clockwise rotation to them or more specifically, $Q_x < 0$ and or $P_y > 0$.

- ***Note that it can be quite difficult to assess curl visually as it does not always look like positive or negative rotation.

- These vector fields appear to rotate counter-clockwise, but the one on left has curl zero, and the one on the right has positive curl.

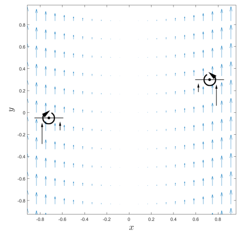


$\mathbf{F} = \langle -y, -x \rangle$

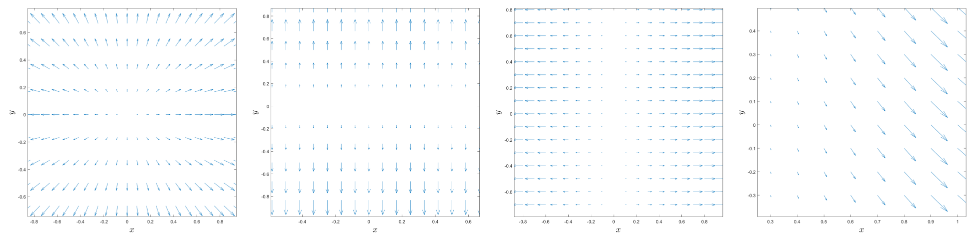


$\mathbf{F} = \langle -y, -0.5x \rangle$

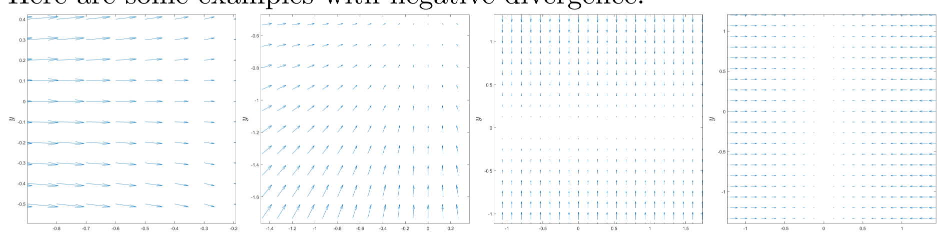
- This vector field has positive curl for $x > 0$ and negative curl for $x < 0$.

 $\mathbf{F} = \langle 0, x^2 \rangle$

- Divergence describes the source character of a vector field.
  Generally, if you look follow along the vector's forward direction, if they get longer, then there is positive divergence or source behavior, and if they get shorter, then there is negative divergence of sink behavior.
  - ➠ Here are some examples with positive divergence:

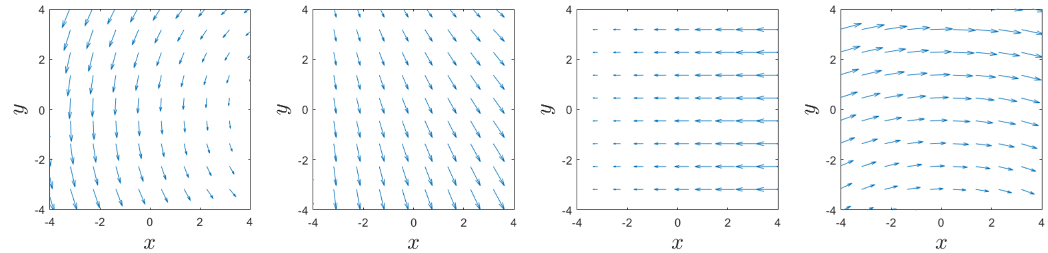
  - ➠ Here are some examples with negative divergence:

  - ➠ ***Note that it can be difficult to assess divergence visually. There are two components, vertical and horizontal. It may be that the point is a source vertically and a sink horizontally or vice versa.
- A vector field that has zero curl in a given region is called *irrotational* on that region
- A vector field that has zero divergence in a given region is called *source-free* on that region
- Divergence of curl is zero: $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ or $\nabla \cdot (\text{curl } \mathbf{F}) = 0$. This means that if we think of the curl as a vector field, it is source-free.
- Curl of a gradient field is zero. $\nabla \times \nabla f = \mathbf{0}$. Note that this is a zero vector. This means that a conservative vector field is irrotational.
- The conditions for curl $\mathbf{F} = \mathbf{0}$ are identical to the conditions for checking if a vector field is conservative. Thus curl $\mathbf{F} = \mathbf{0}$ implies that the vector field is indeed conservative. So conservatives fields are irrotational, and any irrotational vector field is conservative.
- General rotation vector field: $\mathbf{F} = \mathbf{a} \times \mathbf{r}$ has curl given by $\nabla \times \mathbf{F} = 2\mathbf{a}$.

Example problems:

1. Out of the 4 vectors fields below, one has positive curl, one has negative curl, one has positive divergence and one has negative divergence. Identify which is which.



<u>*Solution:*</u>

From left to right:

curl $\mathbf{F} > 0$, as we follow the vectors forward, they turn counter-clockwise
div $\mathbf{F} > 0$, as we follow the vectors forward, they get longer
div $\mathbf{F} < 0$, as we follow the vectors forward, they get shorter
curl $\mathbf{F} < 0$, as we follow the vectors forward, they turn clockwise.

Here are the equations used to generate the images. You are not supposed to be able to figure these out from the graphs, they are just provided for you to confirm the calculations. From left to right:

- $\mathbf{F} = \langle -y, x-7 \rangle$ so 2D curl $\mathbf{F} = 2 > 0$
- $\mathbf{F} = \langle x+5, y-12 \rangle$ so div $\mathbf{F} = 2 > 0$
- $\mathbf{F} = \langle -(x+5), 0 \rangle$ so div $\mathbf{F} = -1 < 0$
- $\mathbf{F} = \langle y+12, -x \rangle$ 2D curl $\mathbf{F} = -2 < 0$

2. Calculate the divergence and curl of $\mathbf{F} = \langle -xy+z, z, y+x \rangle$

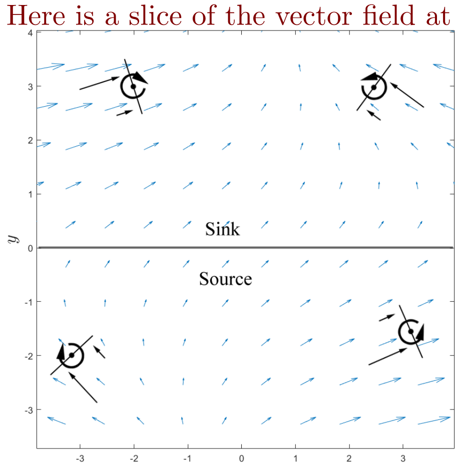
<u>*Solution:*</u>

div $\mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z = -y + 0 + 0 = -y$. This is a dot product, so divergence is a scalar. curl $\mathbf{F} = \nabla \times \mathbf{F} = \langle R_y - Q_z, -(R_x - P_z), Q_x - P_y \rangle = \langle 1-1, -(1-1), 0-(-x) \rangle = \langle 0, 0, x \rangle$

So we can conclude that all points in $\mathbb{R}^3$ with $y < 0$ are sources, and all those with $y > 0$ are sinks.

The general rotational axis of the vector field is $\langle 0, 0, x \rangle$. So when $x > 0$, fixed paddle wheels would rotate counter-clockwise, and when $x < 0$, fixed paddle wheels would rotate clockwise.

Here is a slice of the vector field at $z = 3$:



3. Show that this is a conservative vector field by taking its curl.
$\mathbf{F} = \langle yz, xz, xy \rangle$

Solution:

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \langle x - x, -(y - y), z - z \rangle = \langle 0, 0, 0 \rangle = \mathbf{0}$$