Chapter 14 §6 – Summary and Review (draft: 2019/05/02-15:10:40)

14.6 Surface integrals

Summary of topics and terminology:

- Be able to parameterize a variety of surfaces.
- $z = f(x, y)$ can be parameterized by $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$ and will have upward normal $\mathbf{r}_x \times \mathbf{r}_y = \langle -f_x, -f_y, 1\rangle$.
- Be sure to know when a unit normal is needed and when $\mathbf{r}_u \times \mathbf{r}_v$ can be used unnormalized.
- Tray to practice determining the orientation of a surface be determining on your surface, which direction is increasing u and which direction is increasing v, then u-dir$\times v$-dir will be pointing in a specific direction.
- Be sure to recall all the many important facts about planes, spheres, cylinders, quadric surfaces, etc.
- Surface area: $\iint_S \, dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$
- Scalar surface integral:
  $\iint_S f \, dS = \iint_D f(\mathbf{r}(u, v)) \, |\mathbf{r}_u \times \mathbf{r}_v| \, dA = \iint_D f(\mathbf{r}(u, v)) \, |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$
  D is a domain in the plane of the parametric variables u, v. There is no dot product here, and no vectors at all, just the magnitude of the parametric normal vector.
- if we think of function $f(x, y, z)$ being the area-density (e.g. kg/m^2) then a surface integral gives the mass of the surface.
- Vector surface integral:
  $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv$
  Note that it is important to have the surface oriented correctly here. Again, D is a domain in the plane of the parametric variables u, v.
- The integral of a vector field over an oriented surface gives the flux of the vector field through the surface.

Example problems:

1. Parameterize the surface given by $z = xy^2$ and find its normal vectors.

   Solution:

   $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle = \langle x, y, xy^2\rangle$ will have upward normal $\mathbf{r}_x \times \mathbf{r}_y = \langle -y^2, -2xy, 1\rangle$ and downward normal $\mathbf{r}_x \times \mathbf{r}_y = \langle y^2, 2xy, -1\rangle$. These are not necessarily unit normals.

2. Parameterize the radius 2 sphere centered at the origin and find its normal vectors.

   Solution:

   $\mathbf{r}(\phi, \theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi\rangle$ will have outward normal

   $\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle 4\sin\phi\cos\theta, 4\sin\phi\sin\theta, 4\cos\phi\rangle$

   and inward normal $-\mathbf{r}_\phi \times \mathbf{r}_\theta$. These are definitely not unit normals since $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = 4\sin\phi$.

   Note that for a sphere of radius a, we have $x^2 + y^2 + z^2 = a^2$, and that the gradient of this (if we think of a^2 as a function of x, y, z) is perpendicular to the surface: $\langle 2x, 2y, 2z\rangle$. This gradient points in the direction of steepest increase of a^2. Note that this is indeed parallel to the parametric normal vector it does happen to be the same length! (once you plug in the spherical coordinates for x, y, z)

3. Calculate the surface area of the part of the plane in the positive octant (x, y, z all ≥ 0): $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

   Solution:

   Surface area is given by the scalar surface integral:

   $\iint_S \, dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv$

   So we just need to parameterize our surface and calculate the normal vector using that.

   $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle = \langle x, y, c(1 - \frac{x}{a} - \frac{y}{b})\rangle$ will have upward normal $\mathbf{r}_x \times \mathbf{r}_y = \langle \frac{c}{a}, \frac{c}{b}, 1\rangle$. So $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + \frac{c^2}{a^2} + \frac{c^2}{b^2}}$.

   Thus the surface area is

   $\iint_S \, dS = \int_0^b \int_0^{a(1-\frac{y}{b})} \sqrt{1 + \frac{c^2}{a^2} + \frac{c^2}{b^2}} \, dx \, dy = \sqrt{1 + \frac{c^2}{a^2} + \frac{c^2}{b^2}} \int_0^b a(1 - \frac{y}{b}) \, dy = \sqrt{1 + \frac{c^2}{a^2} + \frac{c^2}{b^2}} a\frac{b}{2}$

   This simplifies to $\frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + a^2c^2}$

4. Find the mass of the cylindrical shell $x^2+y^2 = 25$ for $0 \leq z \leq 3$ with density $f(x, y, z) = z+1$.

   Solution:

   This cylindrical shell is parameterized by $\mathbf{r}(\theta, z) = \langle 5\cos\theta, 5\sin\theta, z\rangle$.

   $\iint_S f \, dS = \iint_D f(\mathbf{r}(\theta, z)) \, |\mathbf{r}_\theta \times \mathbf{r}_z| \, dA = \iint_D f(5\cos\theta, 5\sin\theta, z) \, |\langle 5\cos\theta, 5\sin\theta, 0\rangle| \, d\theta \, dz$

   This becomes $\int_0^3 \int_0^{2\pi} 5(z + 1) \, d\theta \, dz = 45\pi + 30\pi = 75\pi$.

   Notice that for circle (or cylinder) $x^2+y^2 = r^2$, the outward normal is any multiple of $\langle x, y\rangle$.

5. Find the flux of $\mathbf{F} = \langle x, y, 1\rangle$ across surface the cylinder in the previous problem, oriented with outward normals.

   Solution:

   $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(\theta, z)) \cdot (\mathbf{r}_\theta \times \mathbf{r}_z) \, dA$

   Note that the right-hand rule for θ and z makes this normal vector pointing outward.

   $\iint_D \mathbf{F}(\mathbf{r}(\theta, z)) \cdot (\mathbf{r}_\theta \times \mathbf{r}_z) \, dA = \int_0^3 \int_0^{2\pi} \langle 5\cos\theta, 5\sin\theta, 1\rangle \cdot \langle 5\cos\theta, 5\sin\theta, 0\rangle \, d\theta \, dz = 75$