MATH 259 – QUIZ 8 – SPRING 2019 || 25 POINTS || 20 MINUTES

Instructions: Show all work. No collaboration or references. No computational devices allowed without instructor permission.

Print Name ______________________________

1. (5 pts) Calculate the curl and divergence of $\mathbf{F} = \langle x - y, zy, x + y \rangle$.

Solution:

curl $\mathbf{F} = \nabla \times \mathbf{F} = \langle 1 - y, -(1-0), 0 - (-1) \rangle = \langle 1 - y, -1, 1 \rangle$
div $\mathbf{F} = \nabla \cdot \mathbf{F} = 1 + z + 0$

2. (5 pts) Calculate the flux of $\mathbf{F} = \langle 3x, 2y, z \rangle$ through the surface $x + \frac{y}{2} + \frac{z}{2} = 1$, $x \geq 0$, $y \geq 0$, $z \geq 0$ oriented with upward normals.

Solution:

Our surface is $z = f(x, y) = 2 - 2x - y$ and thus the parameterization is $\mathbf{r}(x, y) = \langle x, y, 2 - 2x - y \rangle$ with normal vector $\mathbf{r}_x \times \mathbf{r}_y = \langle -f_x, -f_y, 1 \rangle = \langle 2, 1, 1 \rangle$.

In the xy-plane, our region of integration is the triangle bounded by $x =$, $y = 0$ and $y = 2 - 2x$.

The flux integral is

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{r}(x,y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y)\ dA \\
&= \int_0^1 \int_0^{2-2x} \langle 3x, 2y, 2 - 2x - y \rangle \cdot \langle 2, 1, 1 \rangle\ dy\ dx \\
&= \int_0^1 \int_0^{2-2x} (6x + 2y + 2 - 2x - y)\ dy\ dx \\
&= \int_0^1 \int_0^{2-2x} (4x + y + 2)\ dy\ dx \\
&= \int_0^1 \left[2(2x+1)y + \frac{1}{2}y^2 \right]_0^{2-2x}\ dx \\
&= \int_0^1\ 2(2x+1)(2-2x) + \frac{1}{2}(2-2x)^2\ \ dx \\
&= \int_0^1\ 4(2x+1)(1-x) + \frac{4}{2}(1-x)^2\ \ dx \\
&= \int_0^1\ (1-x)[4(2x+1) + 2(1-x)]\ \ dx \qquad \text{(w/ slick factoring out of } (1-x)) \\
&= \int_0^1\ (1-x)6(1+x)\ \ dx \\
&= \int_0^1\ (6 - 6x^2)\ \ dx \\
&= \left[6x - 2x^3\right]_0^1 = 4
\end{aligned}$$

3. (5 pts) Use Green's theorem to calculate $\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle -y, x + y \rangle$ and $R = [0, 2] \times [0, 1]$.

Solution:

By Green's theorem:

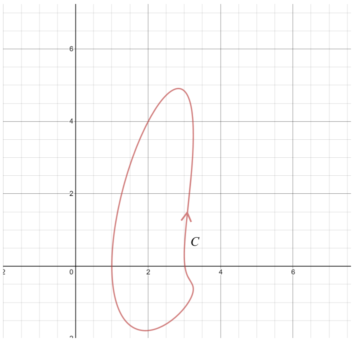
$$\oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \iint_R (Q_x - P_y)\ dA = \iint_R (1 + 1)\ dA = \int_0^1 \int_0^2\ 2\ dx\ dy = 2 \cdot A(R) = 2 \cdot 2 \cdot 1 = 4$$

4. (5 pts) Calculate:

a) $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle 2x + y, 3y^2 + x \rangle$ and

b) $\oint_C \mathbf{G} \cdot \mathbf{n}\, ds$ for $\mathbf{G} = \langle x^2 + y, -2xy \rangle$

where C is the curve shown in the figure below.

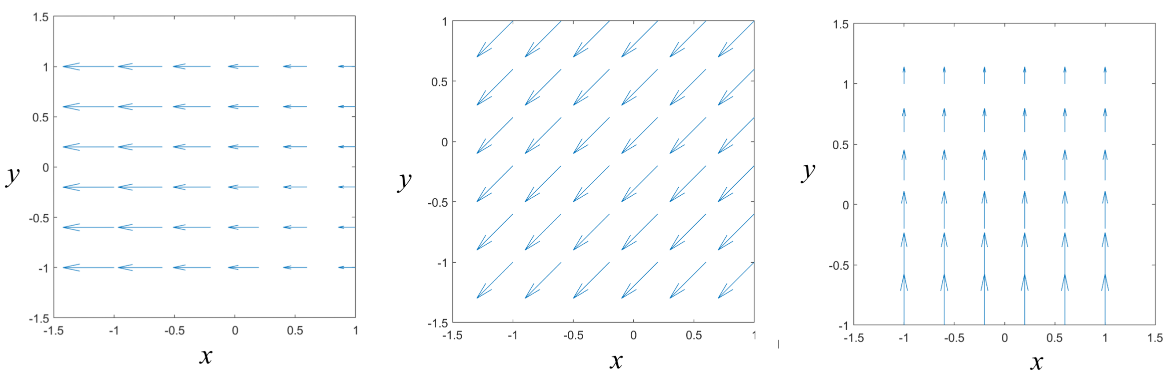


Solution:

a) The 2D curl of $\mathbf{F} = \langle 2x + y, 3y^2 + x \rangle$ is $Q_x - P_y = 1 - 1 = 0$. Thus this vector field is irrotational. By Green's theorem, $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (Q_x - P_y)\, dA = 0$

b) The 2D divergence of $\mathbf{G} = \langle x^2 + y, -2xy \rangle$ is $P_x + Q_y = 2x + (-2x) = 0$. Thus this vector field is source-free. By Green's theorem, $\oint_C \mathbf{G} \cdot \mathbf{n}\, ds = \iint_D (P_x + Q_y)\, dA = 0$

5. (5 pts) Label the vector fields in the figures below as having negative, zero, or positive divergence.



Solution:

Going left to right:

1) The vectors have zero vertical component and the horizontal component is negative. As we move right (in positive x direction) the horizontal component gets shorter. This means it increases (gets less negative). So $Q_x > 0$. The divergence is $Q_x + P_y = (\text{positive}) + 0$.

If we follow the vectors forward, then they get longer, and that is a signal of positive divergence (be careful, as this doesn't absolutely prove it, but it does for a constant direction vector field.

2) These vectors have both constant length and constant direction, so they have both zero curl and zero divergence.

3) These have zero horizontal component, thus $Q = 0$ and $Q_x = 0$. We can see that $P > 0$ and as we move int eh positive y direction, they get shorter, thus $P_y < 0$. The divergence is $Q_x + P_y = 0 + (\text{negative})$.

6. (5 pts) Calculate the surface area of the right-circular cone with height and base radius both equal to 5 (with open base, so just the conical shell). Also find both inward and outward normal vectors.

Solution:

We can use the cone $z^2 = x^2 + y^2$ and let $0 \leq z \leq 5$. This is the upper inverted cone. We parameterize this with cylindrical coordinates $x = z\cos\theta$, $y = z\sin\theta$, and $z = z$. So our parametric surface is $\mathbf{r}(\theta, z) = \langle z\cos\theta, z\sin\theta, z \rangle$.

$\mathbf{r}_\theta \times \mathbf{r}_z = \langle -z\sin\theta, z\cos\theta, 0 \rangle \times \langle \cos\theta, \sin\theta, 1 \rangle = \langle z\cos\theta, z\sin\theta, -z \rangle$
The magntude of which is $|\langle z\cos\theta, z\sin\theta, -z \rangle| = z\sqrt{2}$
So we just integrate this magnitude on the surface of the cone: $\iint_S dS = \int_0^{2\pi} \int_0^5 z\sqrt{2}\, dz\, d\theta = 25\pi\sqrt{2}$

The increasing θ direction goes around counter clockwise and the increaseing z direction is upward so crossing $\mathbf{r}_\theta \times \mathbf{r}_z$ gives the outward/downward normal and $-\mathbf{r}_\theta \times \mathbf{r}_z = \mathbf{r}_z \times \mathbf{r}_\theta$ gives the inward/upward normal

7. (5 pts) Evaluate both $\iint_S f dS$ and $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for S the sphere centered at the origin with radius 1 and $f(x, y, z) = x + y + z$ and $\mathbf{F}(x, y, z) = \langle x, yz, xy \rangle$.

Solution:

We'll do the vector surface integral first. Using the divergence theorem, since the spherical shell is a closed surface:

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \text{div } \mathbf{F} \; dV \\
&= \iiint_E \nabla \cdot \mathbf{F} \; dV \\
&= \iiint_E (1 + z + 0) \; dV \\
&= \iiint_E 1 \; dV + \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho \cos\phi \cdot \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta \\
&= \frac{4}{3}\pi + \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^3 \cos\phi \sin\phi \; d\rho \; d\phi \; d\theta \\
&= \frac{4}{3}\pi + 2\pi \cdot \int_0^1 \rho^3 \; d\rho \cdot \int_0^{\pi} \cos\phi \sin\phi \; d\phi \\
&= \frac{4}{3}\pi + 2\pi \cdot \frac{1}{4} \cdot 0 = \frac{4}{3}\pi
\end{aligned}$$

We could have done less work if we realized that $g(x, y, z) = z$ is sort of like an "odd" function in that it is being integrated over a symmetric volume and has equivalent positive and negative parts which cancel out.

Now for the scalar surface integral.

$$\begin{aligned}
\iint_S f \; dS &= \iint_D f(\mathbf{r}(\phi, \theta)) \; |\mathbf{r}_\phi \times \mathbf{r}_\theta| \; dA \\
&= \int_0^{2\pi} \int_0^{\pi} (\sin\phi \cos\theta + \sin\phi \sin\theta + \cos\phi) \sin\phi \; d\phi \; d\theta \\
&= 0
\end{aligned}$$

We can similarly argue that $f(x, y, z) = x + y + z$ has an "odd" like symmetry in that there are equal balance of positive and negative on the surface of the sphere for each variable which cancel out.