Chapter 14 §7,8 – Summary and Review (draft: 2019/05/05-13:33:02)

14.7 Stokes' theorem

Summary of topics and terminology:

- Stoke's theorem:
$$\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$
- Be able to parameterize a variety of surfaces.
- Generally, we use Stoke's theorem to turn one of the integrals into the other. Often, the line integral on the right is easier since it only involves parameterizing a curve which can often be easier than parameterizing a surface. This is not an absolute rule though. Sometimes the surface integral is easier.
- Recall the definition of curl and how to calculate it.
- Recall how to parameterize curves, especially common ones line line segments and circles.

Example problems:

1. Evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle z, x, xyz\rangle$ and S is the surface of the box $[0, 5] \times [0, 1] \times [0, 2]$ but with open top and oriented with inward normals.

   Solution:

   Using Stoke's $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$ we will evaluate the right side instead.

   Since the top is open, our inward orientation is like the standard upward orientation, we'll need to parameterize our piecewise linear boundary curve counter-clockwise relative to the positive z-axis.

   It will have four pieces, starting at the edge over the x axis and going counter-clockwise (which also happens to be our orientation for the curve itself).

   (a) x goes 0 to 5, $y = 0$, $z = 2$, $\mathbf{r}(t) = \langle 5t, 0, 2\rangle$ with $0 \leq t \leq 1$
   (b) $x = 5$, y goes 0 to 1, $z = 2$, $\mathbf{r}(t) = \langle 5, t, 2\rangle$ with $0 \leq t \leq 1$
   (c) x goes 5 to 0, $y = 1$, $z = 2$, $\mathbf{r}(t) = \langle 5 - 5t, 1, 2\rangle$ with $0 \leq t \leq 1$
   (d) $x = 0$, y goes 1 to 0, $z = 2$, $\mathbf{r}(t) = \langle 0, 1 - t, 2\rangle$ with $0 \leq t \leq 1$

   We do this: $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d}{dt}(\mathbf{r}(t))\ dt$

   Here are the four integrals:

   (a) $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 2, 5t, 0\rangle \cdot \frac{d}{dt}(\langle 5t, 0, 2\rangle)\ dt = \int_0^1\ 10\ dt = 10$
   (b) $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 2, 5, 10t\rangle \cdot \frac{d}{dt}(\langle 5, t, 2\rangle)\ dt = \int_0^1\ 5\ dt = 5$
   (c) $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 2, 5 - 5t, 10 - 10t\rangle \cdot \frac{d}{dt}(\langle 5 - 5t, 1, 2\rangle)\ dt = \int_0^1\ -10\ dt = -10$
   (d) $\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle 2, 0, 10(1 - t)\rangle \cdot \frac{d}{dt}(\langle 0, 1 - t, 2\rangle)\ dt = \int_0^1\ 0\ dt = 0$

   Thus when we add the results we get 5.

14.8 Divergence theorem

Summary of topics and terminology:

- Divergence theorem:
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} \; dV$$
- Recall how to compute triple integrals
- Also remember spherical and cylindrical coordinates for triple integrals

Example problems:

1. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F} = \langle x^3, y^3, z^3 \rangle$ and S being the upper hemispherical shell with radius 1 and closed bottom.

   Solution:

   The surface is given by the equations:
   hemispherical shell: $x^2 + y^2 + z^2 = 1$, $z \geq 0$
   bottom: $z = 0$, $x^2 + y^2 \leq 1$.

   Using the divergence theorem $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} \; dV$ we will evaluate the right side instead.

   The divergence is $\nabla \cdot \mathbf{F} = 3x^2 + 3y^2 + 3z^2$

   We will integrate this over the hemispherical volume. Now that this is a triple integral, we move to standard spherical coordinates. Note that we restrict the limits of ϕ since $z \geq 0$ in our volume.

$$\begin{aligned}
\iiint_E \text{div } \mathbf{F} \; dV &= \iint_E (3x^2 + 3y^2 + 3z^2) \; dV \\
&= \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (3\rho^2) \; \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta \\
&= \int_0^1 3\rho^4 \; d\rho \cdot \int_0^{\pi/2} \sin\phi \; d\phi \cdot \int_0^{2\pi} \; d\theta \\
&= \left[\frac{3}{5}\rho^5\right]_0^1 \cdot [-\cos\phi]_0^{\pi/2} \cdot 2\pi \\
&= \frac{6}{5}
\end{aligned}$$