Math 321 – Summer 2019

additional practice for chapter 2, counting

1. Explain the difference between a combination and permutation.

   Solution:

   For a permutation, order matters. For a combination, order does not matter. Combinations are like subsets. Permutations are specific ordered arrangements.

2. Consider an experiment where a coin is flipped 100 times. How many ways are there to get 47 heads?

   Solution:

   This is a standard combination problem $\binom{100}{47}$.

3. Consider a contest where there are 100 people entered and there are 5 prizes. Assume that once a person wins a prize they are no longer eligible for the other prizes. How many ways are there to award the prizes?

   Solution:

   This is a standard permutation problem, permute 5 out of 100, $100!/95!$.

4. Consider a class of 25 students. A group of 5 students is to be selected for a project.

   (a) Out of the class of 25, how many ways are there to select the group of 5?

   Solution:

   From the context it should seem intuitive that this means selection without replacement and that order doesn't matter. The relevant question is only about group membership. So this is a combination, selecting an unordered subset of the 25. The answer is $\binom{25}{5}$.

   (b) Assume that a particular group of 5 has been selected. How many ways are there to assign 5 specific roles, e.g. leader, note taker, data recorder, calculator, materials gatherer?

   Solution:

   No we are only working with a specific subset of 5 students. There are 5 ways to assign the first role, then 4 ways for the next and so on, then multiply all. This is $5! = 120$. It is just a permutation of the 5 students.

   (c) Out of the entire class of 25, how many ways are there to select the group of 5 where each group member will be assigned a specific role?

   Solution:

   Now, we need to select the 5 students and assign the 5 distinct roles to them all in one go. We can multiply the two previous answers to get $\binom{25}{5} \cdot 5!$ or we can just permute 5 out of 25 directly $\frac{25!}{(25-5)!} = 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 = 6{,}375{,}600$.

   (d) Out of the class of 25, how many ways are there to select the group of 5 where only two group members will be assigned specific roles, e.g. leader and secretary, and the other group members will all be assigned the same role, e.g. regular member?

   Solution:

   This can be solved using a multi-combination, $\binom{25}{1,1,3,20}$. This is the same kind of problem where we have 25 letters, but only 4 distinct types (L, S, G×3, N×20 for leader, secretary, member, non-member). This is sometimes referred to as counting distinguishable permutations since we are still permuting 5 objects, but there are only 3 distinct types (the distinct roles).

5. A restaurant menu has 5 choices for appetizer, 2 choices for main course, 4 choices for dessert, and 6 choices for drink. How many distinct complete meals are possible?

   Solution:

This is a standard application of the multiplication rule. A complete meal consists of 1 item from each subcategory. If two meals have any difference whatsoever, we call them distinct. Thus there are $5 \cdot 2 \cdot 4 \cdot 6 = 240$ total distinct possible meals.

6. We have 3 balls (labeled 1 to 3) and 5 bins (labeled 1 to 5). Each ball is to be placed into a bin. We do not care the order the balls are placed into the bins, but we do care about which specific ball is in which specific bin. Assume each bin can hold any number of balls.

   (a) How many ways can all balls be placed into the same bin?

   Solution:

   Exactly 5. This is identical to simply choosing a single bin $\binom{5}{1}$.

   (b) How many ways can each ball be placed into a different bin?

   Solution:

   This is actually identical to permuting 3 out of 5 objects, $5!/2! = 60$. We want to choose 3 of the bins and arrange them in order for balls 1 to 3. We are distinguishing between which ball goes in which bin, e.g. If bins 2, 4, and 5 each get a ball, and the balls in them are 3, 1, and 2, respectively, then this is different to the balls being in order 1, 2, and 3.

   (c) How many ways can all balls be placed into two bins?

   Solution:

   Here are the steps:

   1) Choose the two bins. $\binom{5}{2}$
   2) Decide which bin will have 2 total balls (the other bin will have 1 ball). $\binom{2}{1}$
   3) Decide which 2 balls will go in the bin containing two balls
   (the other ball will automatically go in the other bin). $\binom{3}{2}$

   So the total number of ways to do this is $\binom{5}{2}\binom{2}{1}\binom{3}{2} = 60$.

7. Consider a standard 52 card deck.

   (a) If 5 cards are drawn without replacement, how many ways can they all have the same suit?

   Solution:

   There are 4 suits and 13 cards per suit. We just need to pick a suit and then pick 5 cards from that suit. $\binom{4}{1}\binom{13}{5}$.

   As far as standard poker goes, this includes all flushes, straight flushes, and royal flushes.

   (b) Continuing the above part, how many ways can we pick 5 cards that are all teh same suit and for a straight (consecutive face values)?

   Solution:

   Similar to the previous part, we pick a suit first, but instead of picking 5 cards out of 13, we just need to pick one card. I like to pick the low card, since then next 4 face values are then already determined. So the total number is $\binom{4}{1}\binom{10}{5}$. Note that ace can be high or low, so the low card can be anything from {A, 2, 3, ..., 10} since a 10, J, Q, K, A is the highest straight possible.

   As far as standard poker goes, this includes all straight flushes and royal flushes.

   (c) How many ways can we pick 5 cards where two of them are red and three are black?

   Solution:

   The diamond and heart suits are red, and the club and spade suits are black, thus there are a total of 26 red cards and 26 black cards. We just pick 2 cards form one set and 3 cards from the other set for a total of $\binom{26}{2}\binom{26}{3}$.

8. Consider an urn that contains 15 red balls, 5 green balls, and 10 blue balls.

(a) If we draw 3 balls without replacement, how many ways can they all be red?

Solution:

This is a standard combination problem, $\binom{15}{3}$.

(b) If we draw 3 balls without replacement, how many ways can they be all the same color?

Solution:

We have to work this out carefully. We cannot just simply choose a color and then choose the balls and use the multiplication rule because each color has a different number of balls to choose from. We must count for each color separately then add the results.

$\binom{15}{3} + \binom{5}{3} + \binom{10}{3}$

(c) If we draw 3 balls without replacement, how many ways is it possible to have 1 ball of each color?

Solution:

We just do a "choose one" for each color and multiply: $\binom{15}{1}\binom{5}{1}\binom{10}{1}$.

(d) If we draw 3 balls with replacement, and we care about the order they were drawn in, how many total possible outcomes are there? Treat each ball as distinct. You might imagine that even though many are the same color, that they are labeled with letters or numbers to tell them apart.

Solution:

There are 30 total possible balls for each of the 3 choices so 30^3. Normally, we would think of each of these possibilities as equally likely.

Notice that we are treating each ball as an individual above. This includes multiple ways to have all 3 balls be red for example (the same red ball drawn repeatedly, or maybe just three different red balls drawn, etc.). So if we only care about the ordered colors that are drawn, then we are in effect distinguishing between fewer number of possible outcomes. In this sense there are only 27 *distinct* possibilities since we have three color choices for each draw, 3^3. If we think of the first calculation as being equally likely outcomes,then these 3^3 outcomes are not equally likely.