Chapter 1 Summary and Review (draft: 2019/09/29-12:30:13)

Counting

Summary of topics and terminology:

- Multiplication rule: n_i ways for step $i = 1, 2, ..., k$ then $n = n_1 \cdot n_2 \cdots n_k$ total ways.
- Sampling k times from set of size n with replacement, order matters: n^k
- Permutation: Sampling k times from set of size n without replacement, order matters: ${}_nP_k = \frac{n!}{(n-k)!}$. This is simultaneously drawing a subset of size k and then arranging them in a specific order.
- Combination: Sampling k times from set of size n without replacement, order doesn't matters: ${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$. This is just drawing a subset of size k.
- Distinguishable permutations: $\binom{n}{n_1, n_2, ..., n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}$
This is like drawing multiple disjoint subsets of size $n_1, n_2, ..., n_k$.
- k indistinguishable balls into n distinguishable bins: there are $\binom{k+n-1}{k}$ *distinguishable* ways to arrange the balls in the bins. If we consider each ball equally likely to go into any bin, then this formula does not tell up how many equally likely outcomes there are. There are n^k equally likely outcomes, but many of these will be indistinguishable from one another.
- When selecting a number of properties/objects to allocate in a specific way, we must decide carefully whether to choose simultaneously or in separate steps. If we have 3 dice and want a pair and a single, we choose their face values separately: $\binom{6}{1,1,4}$. If we want 3 singles, then we choose those face values simultaneously $\binom{6}{3,3}$. This is because at the stage of selecting face values we don't yet care to distinguish between which die gets what face value.
- Try to count in a variety of ways, tree diagrams, listing out explicitly, etc.

Example problems:

1. If there are 12 items on a menu, and 3 orders are to be placed, how many possible outcomes are there?

   Solution:

   If we assume there are enough of each item so that anyone can order any item, then there are 12^3 possibilities.

   If we assume there is only 1 of each item, then there are $12 \cdot 11 \cdot 10$ (i.e. ${}_{12}\mathrm{P}_3$) ways.

   If we assume that the 3 orders are all going to the same person and that each order will be for a different item (and we don't care about which order was placed first, etc.), then there are $\binom{12}{3}$ ways.

   It just depends on what we want to distinguish between.

2. For a standard 52 card deck with 13 face values and 4 suits, if 5 cards are drawn without replacement, how many ways can it be two pair where the same two suits appear is both pairs?

   Solution:

   We'll need 3 face values: $\binom{13}{2,1,10}$, 2 for the pairs, 1 for the single, and 10 unused.
   We'll choose our 2 suits for the pairs: $\binom{4}{2}$. We only need to do this once, since each pair

will have the same 2 suits.
Then choose the suit for the single: $\binom{4}{1}$.

So: $P(2 \text{ pair \& 1 single with both pairs having the same 2 suits}) = \frac{\binom{13}{2,1,10}\binom{4}{2}\binom{4}{1}}{\binom{52}{5}}$.

3. Consider the string of letters *PROBABILITY*. If we rearrange the letters, how many distinguishable permutations are there?

   <u>*Solution:*</u>

   $\binom{11}{2,2,1,1,1,1,1,1,1}$

   Note that if we consider each individual letter equally likely to go in any spot, then this distinguishable permutation formula is not necessarily useful for calculating probabilities.

4. If we roll five 6-sided dice, how many distinguishable ways are there to get 2 pair (and 1 single)? How many total ways can 2 pair and 1 single occur?

   <u>*Solution:*</u>

   We simply need to choose the face values that will show: $\binom{6}{2,1,3}$ since we choose 2 face values for the pairs and 1 face value for the single and 3 unused face values. We choose the values for the pairs simultaneously because we don't want to overcount. A pair of 2's with dice A and B and 3's with dice C and D is nondistinguishable from a pair of 3's on A and B and 2's on C and D.

   To count how many distinct physical ways this can occur, we allocate these face values to the 5 dice carefully with $\binom{5}{2,2,1}$.

   So there are $\binom{6}{2,1,3}\binom{5}{2,2,1}$ total physically distinct ways but only $\binom{6}{2,1,3}$ distinguishable ways.