Chapter 2 Summary and Review (draft: 2019/09/29-12:57:02)

Probability

Summary of topics and terminology:

- Outcomes are individual possibilities for a random experiment/process.
- Sample space, S, is the set of all possible outcomes.
- Events are subsets of the sample space. They can be singletons containing a single outcome, contain any number of outcomes even infinitely many, or even empty with no outcomes.
- Simple probability for equally likely outcomes: $P(A) = \frac{|A|}{|S|}$.
- Axioms of probability: $P(S) = 1$, $P(A) \geq 0$, $P(\cup A_i) = \sum P(A_i)$ for disjoint events.
- Misc. rules: $P(\emptyset) = 0$, also $A \subset B \Rightarrow P(A) \leq P(B)$.
- Events A and B are mutually exclusive if they are disjoint (empty intersection): $A \cap B = \emptyset$.
- Events $A_1, A_2, ...$ are called pair-wise mutually exclusive (or pairwise disjoint, or simply a disjoint collection of events) when $A_i \cap A_j = \emptyset$ for all $i \neq j$.
- Inclusion/exclusion, 2 events: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- Inclusion/exclusion, 3 events:
  $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.
- Inclusion/exclusion, n events:
  $P(\cup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i) - \sum_{i \neq j} P(A_i \cap A_j) + \sum_{i,j,k \text{ distinct}} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(\cap_{i=1}^n A_i)$.
- Partition: $\cup A_i = S$, A_i pairwise mutually exclusive (disjoint).
- A and A^c form a partition. $P(A^c) = 1 - P(A)$.
- Conditional probability: $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$
- Conditional probability is a probability and satisfies all axioms and properties of probability. I.e. any equation you see with $P(\cdot)$ you can replace with $P(\cdot \mid A)$.
  E.g. $P(B^c \mid A) = 1 - P(B \mid A)$.
- For any event A and partition B, B^c, $P(A) = P(A \cap B) + P(A \cap B^c)$
  $= P(A \mid B)P(B) + P(A \mid B^c)P(B^c)$
- For partition $\{A_i\}_{i=1}^n$ we have $P(B) = \sum_{i=1}^n P(B \cap A_i)$
- $P(A \cap B) = P(A \mid B)P(B)$.
- $P(A \cap B \cap C) = P(A \mid B \cap C)P(B \mid C)P(C)$. This generalizes to any number of events. This can be understood in the sequential draw context.
- Events A and B are independent if $P(A \cap B) = P(A)P(B)$. This is also equivalent to $P(A|B) = P(A)$ and $P(B|A) = P(B)$.
- Events $\{A_i\}_{i=1}^n$ are called independent only if $P(\cap_{i \in I} A_i) = \prod_{i \in I} P(A_i)$ for every set of subindexes I. Pairwise independence is not enough.
- Make sure to not confuse “independence” with “mutually exclusive”.

- Baye's theorem: $P(A \mid B) = \dfrac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)}$
- Baye's theorem: $P(A_i \mid B) = \dfrac{P(B \mid A_i)P(A_i)}{\sum_{i=1}^{n} P(B \mid A_i)P(A_i)}$ where $\{A_i\}_{i=1}^{n}$ is a partition of the sample space.

Example problems:

1. If a fair coin is flipped 5 times, what is the probability that there is at least one head?

   Solution:

   $P(\text{at least one}) = 1 - P(\text{none})$ is a very useful thing to keep in mind.

   $P(\text{no heads}) = \frac{1}{2^5}$ thus $P(\text{at least 1 H}) = 1 - \frac{1}{2^5}$.

2. Consider an urn that contains 6 red, 3 blue, 4 green, and 2 yellow balls. If 4 are drawn without replacement, find the probability that the 1st is yellow, the 2nd is red, the 3rd is green, and the 4th is red.

   Solution: Let $R_i = i^{th}$ is red, and similarly construct events for other colors.

   $P(Y_1 \cap R_2 \cap G_3 \cap R_4) = P(Y_1)P(R_2 \mid Y_1)P(G_3 \mid Y_1 \cap R_2)P(R_4 \mid Y_1 \cap R_2 \cap G_3) = \frac{2}{15} \cdot \frac{6}{14} \cdot \frac{4}{13} \cdot \frac{5}{12} = \frac{2}{3 \cdot 7 \cdot 13}$

3. Consider the same problem above, but now calculate the probability the draw has 1 yellow, 1 green, and 2 red.

   Solution:

   From the text you should infer that this is an unordered draw. So we can use combinations.

   $$P(2R, 1G, 1Y) = \frac{\binom{6}{2}\binom{3}{0}\binom{4}{1}\binom{2}{1}}{\binom{15}{4}} = \frac{\frac{6 \cdot 5}{2} \cdot 4 \cdot 2}{15 \cdot 14 \cdot 13 \cdot 12} = \frac{1}{3 \cdot 7 \cdot 13}$$

4. Consider the same setup above but now draw 8 balls w/o replacement. What is the probability that there are 3 red, 2 blue, 2 green, and 1 yellow?

   Solution:

   $$P(3R, 2B, 2G, 1Y) = \frac{\binom{6}{3}\binom{3}{2}\binom{4}{2}\binom{2}{1}}{\binom{15}{8}}$$

   Another way to work this is to treat it as a sequential draw problem, but then account for all possible orderings which amount to "distinguishable permutations".

   $$P(3R, 2B, 2G, 1Y) = \underbrace{\frac{6}{15}\frac{5}{14}\frac{4}{13}}_{\text{red}} \cdot \underbrace{\frac{3}{12}\frac{2}{11}}_{\text{blue}} \cdot \underbrace{\frac{4}{10}\frac{3}{9}}_{\text{green}} \cdot \underbrace{\frac{2}{8}}_{\text{yel.}} \cdot \binom{8}{3,2,2,1}$$

   The sequential draw product of fractions on the left should be intuitive to understand. The multi-combination on the right is for choosing which draws will be which color. This increases the count above the ordered draw case since we don't care about which order the balls were drawn in here, just that we get the required number for each color.

5. Consider the experiment of rolling eight 6-sided dice. Calculate the probability of (a) 4 pairs, (b) two quadruples, (c) 2 triples and 1 pair.

   Solution:

   (a) $P(\text{4 pairs}) = \dfrac{\binom{6}{4,2}\binom{8}{2,2,2,2}}{6^8}$

   (b) $P(\text{2 quadruples}) = \dfrac{\binom{6}{2,4}\binom{8}{4,4}}{6^8}$

(c) $P(2 \text{ triples } \& \text{ 1 pair}) = \dfrac{\binom{6}{2,1,3}\binom{8}{3,3,2}}{6^8}$

6. Consider a store closet containing 1,000 circuit boards 500 of which are type A, 400 are type B, and 100 are type C. Type A , B, and C boards are faulty with probability 0.002, 0.005, and 0.01 respectively. If a circuit board is randomly selected and found to be faulty, what is the probability it is type A?

   Solution:

   We are assuming that each individual circuitboard is equally likely to be selected and one is randomly selected. We have that:
   $P(F|A) = 0.001$ and $P(A) = 1/2$
   $P(F|B) = 0.005$ and $P(B) = 2/5$
   $P(F|C) = 0.01$ and $P(C) = 1/10$

   We wish to calculate $P(A|F)$. We use Baye's theorem

$$\begin{aligned} P(A|F) &= \frac{P(F|A)P(A)}{P(F|A)P(A) + P(F|B)P(B) + P(F|C)P(C)} \\ &= \frac{(0.002)(0.5)}{(0.002)(0.5) + (0.005)(0.4) + (0.01)(0.1)} \\ &= \frac{10}{10 + 20 + 10} = \frac{1}{4} \end{aligned}$$

7. If 15 people are to sit in 15 chairs arranged in a circle, what is the probability that a particular pair will en up seated in two adjacent chairs? What is the probability that a particular group of 3 will be seated in a contiguous block of 3 chairs?

   Solution:

   For the pair, we just need to choose two adjacent chairs and then account for assigning the two people to the chairs. Here we choose a chair for one person then choose one of the adjacent chairs for the other person: $\frac{\binom{15}{1}\binom{2}{1}}{15 \cdot 14}$

   For the trio, we need to choose a block of 3 chairs and then arrange the people in them: $\frac{\binom{15}{1}3!}{15 \cdot 14 \cdot 13}$

8. Consider the word *PROBABILITY*. If we randomly rearrange the letters what is the probability that the two B's will end up next to each other?

   Solution:

   There are 11 letters and thus 10 pairs of adjacent spaces, so there are $\binom{10}{1}$ ways to choose two adjacent spots for the B's and then 2! ways to arrange the B's next to each other. Then there are 9! ways to arrange the rest of the letters. There are a total of 11! ways to rearrange all of the letters. Thus the probability sought is $\frac{10 \cdot 2 \cdot 9!}{11!} = \frac{2}{11}$.

   Another possibly more intuitive way to work this out is to just arrange the B's in the first two spots: $2 \cdot 1 \cdot 9!$ and divide this by 11! but then multiply by $\binom{10}{1}$ to allow the B's to then be in any adjacent spots.

   Note that this is different from the distinguishable permutations method. Even though the B's are indistinguishable from each other, we still need to count all ways they can be ordered since they are physically distinct objects. If we were to simply count how many distinguishable permutations there were with adjacent B's, then there are $\binom{10}{1} \cdot \binom{9}{1,1,1,1,2,1,1,1}$ of those. For this we do not distinguish between the B's or I's, so we are counting fewer distinguishable arrangements than in the case where we are calculating probability.