Math 421 – FALL 2019 – Special Discrete Distributions - Friday, 10/18 - Due: Wednesday, 10/23

Name:____

Instructions: Write answers on this sheet, but use scratch paper to work out solutions. Read sections §5.1 – 7.

A **Bernoulli** random variable can be used to model a random experiment with only two possible outcomes.

Definition. A **Bernoulli** discrete RV X with success probability θ has probability distribution given by

$$f(x) = \begin{cases} 1-\theta & \text{for } x = 0 \\ \theta & \text{for } x = 1 \end{cases}$$

and is denoted by $X \sim \text{Bernoulli}(\theta)$. $X = 1$ is interpreted as a success, and $X = 0$ is interpreted as a failure.

Note: Read $X \sim \text{Bernoulli}(\theta)$ as "X is Bernoulli distributed with parameter θ."

1. Calculate the mean and variance of X if $X \sim \text{Bernoulli}(\theta)$.

2. If X_1 and X_2 are independent and both Bernoulli with probability of success θ, find their joint probability distribution.

3. Consider $X_1, X_2, \ldots, X_n$ are independent and each Bernoulli with probability of success θ.

a) Interpret $\sum_{i=1}^{n} X_i$ in terms of the context of the experiment. *(Hint: what are the numerical values of each X_i and thus how can we interpret their sum?)*

b) Find the probability distribution of $X = \sum_{i=1}^{n} X_i$. *(Hint: use exponent properties such as $\theta^{x_i}\theta^{x_j} = \theta^{x_i + x_j}$, and combinations to choose which X_i are successes or failures.)*

Definition. A **Binomial** discrete RV X with n independent Bernoulli trials and success probability θ has probability distribution given by

$$f(x) = b(x; n, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

and is denoted by $X \sim \text{Bin}(n, \theta)$ or $X \sim \text{B}(n, \theta)$. $X =$ the number of successes.

4. Use the expected value of summation (Theorem 4.14, p. 135 in §4.7) to find $\text{E}(X)$ and $\text{Var}(X)$.

5. **Geometric:** What if we perform Bernoulli trials with probability of success θ until we get a success? Let X represent the total number of trials. The minimum value of X is one, since we stop if the first trial is a success, but there if no limit on how many trials it could take. Find the probability distribution of X. This is called a **Geometric** random variable. It's probability function is denoted $b^*(x; k, \theta)$ and denoted $X \sim \text{Geom}(\theta)$.
(Hint: which trial is a success?)

6. For $X \sim \text{Geom}(\theta)$, find $\text{E}(X)$ and $\text{Var}(X)$. Use the following facts:
$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ for $|r| < 1$
$\sum_{n=1}^{\infty} nr^{n-1} = \frac{d}{dr} \sum_{n=1}^{\infty} r^n$.
$\sum_{n=1}^{\infty} n^2 r^{n-1} = \frac{d}{dr} \left(r \cdot \frac{d}{dr} \sum_{n=1}^{\infty} r^n \right)$.
(Hint: be careful about the starting index before using the geometric series formula and let $r = 1 - \theta$.)

7. **Negative Binomial:** Similar to the previous problem, we now perform Bernoulli trials until we get k total successes. Again let X represent the total number of trials. The minimum value of X is k, since we stop if the first k trials are all successes, but again there if no limit on how many trials it could take. Find the probability distribution of X. This is called a **Negative Binomial** random variable. It's probability function is denoted $g(x; \theta)$ and is denoted as $X \sim \text{NegBin}(k, \theta)$ or $X \sim \text{NB}(k, \theta)$.
(Hint: you will need to think carefully about choosing which trials are allowed to be successes.)

Definition. A **Poisson** discrete RV X with rate parameter λ has probability distribution given by

$$f(x) = p(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad \text{for } x = 0, 1, 2, \ldots$$

and is denoted by $X \sim \text{Pois}(\lambda)$. $X =$ the number of events.

8. Find the moment generating function for X given that $X \sim \text{Pois}(\lambda)$ and use it to find $\text{E}(X)$ and $\text{Var}(X)$.

9. **Interpretation of the Poisson RV.** Consider an experiment where events occur randomly over time, but that the expected number of events during any given time period is proportional to the length of time, and the number of events in disjoint time intervals are independent. The rate parameter λ can be thought of as the number of events per unit time.

a) **Scaling the Poisson rate parameter:** If vehicles cross a particular point on a roadway at a rate of λ vehicles per hour, let X be the RV representing the number of vehicles that cross this point during a t hour time period. Find the probability distribution for X.

b) If vehicles arrive at rate 15 per hour, find the probability that no vehicles arrive during a 25 minute interval.