421 - Q2 soln.

1) $E(X \mid X > t) = \dfrac{\int_t^\infty x \cdot \alpha e^{-\alpha x}\,dx}{\int_t^\infty \alpha e^{-\alpha x}\,dx} = \dfrac{[-x e^{-\alpha x}]_t^\infty - \int_t^\infty e^{-\alpha x}\,dx}{\cancel{0} + e^{-\alpha t}}$

$= \dfrac{t e^{-\alpha t} + \frac{1}{\alpha} e^{-\alpha t}}{\cancel{0} + e^{-\alpha t}}$

$= t + \dfrac{1}{\alpha}$

2) $E(X^2) = \int_0^1 \int_0^1 x^2 \cdot \cancel{x}\, \frac{1}{3}(4x + 3y^2)\,dx\,dy$

$= \frac{1}{3} \cdot [x^4]_0^1 + \left[\frac{x^3}{3}\right]_0^1 \cdot \left[\frac{y^3}{3}\right]_0^1 = \frac{1}{3} + \frac{1}{9} = \boxed{\frac{4}{9}}$

3) $E(X) = \cancel{0 \cdot (0.3) + 1 \cdot (0.6 - p)} = 0.6$

$E(Y) = \cancel{0 \cdot (0.1) + 1 \cdot (p)} = 0.1 + p$

$E(XY) = 1 \cdot p$

$Cov(X,Y) = E(XY) - \mu_X \cdot \mu_Y \cancel{= p - p(0.6 - p)}$

$\cancel{= -0.4p + p^2}$

$\cancel{= 0 \Rightarrow p = 0, -0.4}$

$= p - 0.6(0.1 + p)$

$= \cancel{0}\; 0.4p - 0.06 \Rightarrow p = \frac{0.06}{0.4}$

$= \frac{6}{40} = \frac{3}{20}$

4) $Cov(aX_1 + bX_2, cX_1 + dX_2) = ac \cdot Var(X_1)$

$+ bd\, Var(X_2)$

$+ (ad + cb) \cdot Cov(X_1, X_2)$

$a = 1, b = 1$

$c = 1, d = -1$

$Cov(X_1, X_2) = 0$ b/c indep.

$= \boxed{\frac{1}{\alpha_1^2} - \frac{1}{\alpha_2^2}}$

$Var(X_1) = \frac{1}{\alpha_1^2}$
$Var(X_2) = \frac{1}{\alpha_2^2}$
} b/c Exp. RVs

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5) $M(t) = (1-t\beta)^{-\alpha}$

$M'(t) = +\alpha\beta(1-t\beta)^{-\alpha-1}$

$M''(t) = \alpha\beta^2(\alpha+1)(1-t\beta)^{-\alpha-2}$

$M'(0) = \alpha\beta$ $\Rightarrow E(X) = m_1 = \boxed{\alpha\beta}$

$M''(0) = \alpha\beta^2(\alpha+1)$ $\quad E(X^2) = m_2 = \alpha\beta^2(\alpha+1)$

$Var(X) = E(X^2) - \mu^2$

$= \alpha\beta^2(\alpha+1) - \alpha^2\beta^2$

$= \boxed{\alpha\beta^2}$

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6) a)

$X \sim Geom(\theta = 0.05)$

$P(X \le 4) = \sum_{k=1}^{4}(0.05)(0.95)^{k-1} = (0.05)\left[\cancel{0.95+(0.95)} \; 1+0.95+0.95^2+0.95^3\right]$

$\cong 0.1855 \approx 19\%$

b) $X \sim Bin(n=10, \theta=0.05)$

$P(X=3) = \binom{10}{3}(0.05)^3(0.95)^7 \approx 0.010475$

$\approx 1\%$

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7) a) $X \sim Pois(\lambda = 20 \cdot 12 = 240)$ or $X \sim Gamma(\alpha = \text{shape} = 226, \beta = \text{scale} = \frac{1}{240})$

$P(X \le 225) = \sum_{k=0}^{225} \frac{e^{-240}(240)^k}{k!}$ $\quad P(X \ge 1) = \int_1^\infty f(x)dx$

≈ 0.1749545 $\quad = \cancel{\int_1^\infty \frac{1}{\Gamma(226)} \cdots e}$

$=$

225 or less in 1 yr $\Rightarrow 226^{th}$ obs. is after 1 yr.

$\Rightarrow$ 1 yr or longer to get 226

b) $X \sim Gamma(\text{shape} = 300, \text{scale} = \frac{1}{20})$ (α, β)

$E(X) = \alpha\beta = \frac{300}{20} = \boxed{15 \text{ months}}$