Chapter 6 Summary and Review (draft: 2019/11/17-12:47:01)

Uniform, exponential, gamma

Summary of topics and terminology:

- Uniform: $X \sim \mathrm{U}[a,b]$. We can think of every real number in $[a,b]$ as "equally likely".
  - ➠ $f_X(x) = \frac{1}{b-a}$. pdf, continuous, $x \in [a,b]$.
  - ➠ $\mathrm{E}(X) = \frac{a+b}{2}$
  - ➠ $\mathrm{Var}(X) = \frac{1}{12}(b-a)^2$
- Exponential: $X \sim \mathrm{Exp}(\text{rate} = \alpha)$ or $X \sim \mathrm{Exp}(\text{mean} = \theta)$. If events are occurring at random times of locations over some continuous physical extent, then the time or space between events in modeled by an exponential random variable. E.g. events occur over time, with θ being the average time elapsed between events or α being the number of events per unit time (rate of events per unit time). If events occur over a spatial length (like imperfections on rope), then θ is the average length between events, and α is the average number of events per unit length.
  - ➠ $f_X(x) = \alpha e^{-\alpha x}$ or $f_X(x) = \frac{1}{\theta}e^{-\frac{1}{\theta}x}$. pdf, continuous, $x > 0$.
  - ➠ $\mathrm{E}(X) = \frac{1}{\alpha} = \theta$
  - ➠ $\mathrm{Var}(X) = \frac{1}{\alpha^2} = \theta^2$
  - ➠ Memoryless property: $\mathrm{P}(X > a + t \mid X > t) = \mathrm{P}(X > a)$.
  - ➠ $M_X(t) = (1-\theta t)^{-1}$ or $M_X(t) = (1 - \frac{1}{\alpha}t)^{-1}$
- Gamma: $X \sim \mathrm{Gamma}(\text{shape} = \alpha, \text{scale} = \beta)$. Similar tot eh exponential, events are occurring at random times over an interval. If we wish to ask how long it takes for a fixed number of events to occur, then we model that time with a gamma distributed random variable. The **shape** parameter is the number of events we want to observe, and the **scale** parameter is the average time between events. This is not the only use or interpretation of the gamma distribution though.
  - ➠ $f_X(x) = \frac{1}{\Gamma(\alpha)}\frac{1}{\beta^\alpha}x^{\alpha-1}e^{-\frac{1}{\beta}x}$. pdf, continuous, $x > 0$.
  - ➠ $\mathrm{E}(X) = \alpha\beta$
  - ➠ $\mathrm{Var}(X) = \alpha\beta^2$
  - ➠ $M_X(t) = (1-\beta t)^{-\alpha}$.
- Relationship between Poisson, Exponential, and Gamma distributions. For events that occur randomly over a continuous physical extent (e.g. interval of time or length) at rate $\alpha =$ the number of events per unit of physical extent, then
  - ➠ The number of events over t units of time is Poisson with parameter $\lambda = \alpha t$: $f_X(x) = e^{-\alpha t}\frac{(\alpha t)^x}{x!}$ for $x = 0, 1, 2, \ldots$.
  - ➠ The time between events in Exponential with rate parameter α or mean value $\theta = \frac{1}{\alpha}$. $f_X(x) = \alpha e^{-\alpha t}$ or $f_X(x) = \frac{1}{\theta}e^{-\frac{1}{\theta}t}$ for $x > 0$
  - ➠ The time elapsed for k events to occur is Gamma with shape parameter k and scale parameter $\theta = \frac{1}{\alpha}$. $f_X(x) = \frac{1}{\Gamma(k)}\alpha^k x^{k-1}e^{-\alpha x}$ or $f_X(x) = \frac{1}{\Gamma(k)}\frac{1}{\theta^k}x^{k-1}e^{-\frac{1}{\theta}x}$ for $x > 0$

Example problems:

1. For X uniformly distributed on $[a, b]$, calculate $\mathrm{E}(X \mid X > t)$.

   Solution:

   $\mathrm{E}(X \mid X \geq t) = \frac{\int_t^b x \frac{1}{b-a} \; dx}{\int_t^b \frac{1}{b-a} \; dx} = \frac{1}{2} \frac{b^2-t^2}{b-t} = \frac{1}{2}(b+t) \quad$ for $a \leq t \leq b$.

   Think about this. The non-conditional average $\mathrm{E}(X) = \frac{a+b}{2}$ is the the midpoint of the interval. So the conditional average is still the midpoint between the lowest and highest possible values.

2. If the usable lifespan for a lightbulb can be modeled by an exponential distribution with mean lifespan 8000 hours, find the probability that the bulb lasts longer than 9000 hours.

   Solution:

   We have parameter $\theta = 8000$ and pdf $f(x) = \frac{1}{8000} e^{-\frac{1}{8000}x}$. We can think of $\alpha = \frac{1}{8000}$ as the rate that "end of life" events occur for the lightbulb.

   $$P(X > 9000) = \int_{9000}^{\infty} \frac{1}{8000} e^{-\frac{1}{8000}x} dx = -e^{-\frac{1}{8000}x}\Big|_{9000}^{\infty} = e^{-\frac{9}{8}} \approx 0.3246525$$

3. Consider a large array of LEDs (light-emitting electronic components) where failures occur on average 1 per year. The array will need to be completely replaced once two such failures have occurred. Calculate the probability that the LED array last longer than 3 years.

   Solution:

   We are asking for the time to observe two LED failures to take longer than 3 years. The time to observe 2 failures is modeled by a Gamma distributed random variable with shape$= 2$ and scale$= 1$. In the α & β definition of the gamma, this is $\alpha = 2$, $\beta = 1$. Just to be sure, this sets our timescale to be measured in years. If we were to measure time in months, then $\beta = 12$ would be the scale parameter.

   The pdf is then $f_X(x) = \frac{1}{\Gamma(2)} x^{2-1} e^{-x} = x e^{-x}$ for $x > 0$.

   The RV X here is how much time elapses to observe two total LED failures. We want to know the probability this takes 3 years or longer.

   $P(X > 3) = \int_3^{\infty} x e^{-x} dx$.

   Using integration-by-parts: $u = x, du = dx, dv = e^{-x} dx, v = -e^{-x}$ gives
   $[-xe^{-x}]_3^{\infty} + \int_3^{\infty} e^{-x} dx = 3e^{-3} + e^{-3} = 4e^{-3} \approx 0.1991483$

4. In the same LED array problem above, calculate the probability that example two failures occur in a 3 year period.

   Solution:

   Here we are asking the probability of a specific number of events over a fixed length of time. We will use Poisson here. Let X be the number of failures in a 3 year time period. Here we will disregard the fact that after 2 failures the LED array will be replaced. Assume it will be in operation indefinitely so that LED failures can continue to occur.

   $X \sim \text{Pois}(\lambda = 3)$ since on average one failure occurs per year, we expect 3 failures to occur over the 3 year period.

   $\mathrm{P}(X = 2) = \frac{e^{-3} 3^2}{2!} = \frac{9}{2} e^{-3} \approx 0.2240418$

Normal

Summary of topics and terminology:

- Normal: $X \sim \mathrm{N}(\mu, \sigma^2)$. The normal distribution is arguably the most important distribution in all of probability and statistics.
- $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$. pdf, continuous, $-\infty < x < \infty$.
- $\mathrm{E}(X) = \mu$
- $\mathrm{Var}(X) = \sigma^2$
- $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$
- Central moments: $\mathrm{E}[(X-\mu)^n] = (n-1)!!\sigma^n$ for n-even and $\mathrm{E}[(X-\mu)^n] = 0$ for n-odd, where $(n-1)!!$ is the double factorial which is the product of all odd numbers from n on down to 1, or the product of all even numbers from n on down to 2.
- 68-95-99.7 rule:
  - $\mathrm{P}(|x-\mu| \leq \sigma) \approx 0.68$
  - $\mathrm{P}(|x-\mu| \leq 2\sigma) \approx 0.95$
  - $\mathrm{P}(|x-\mu| \leq 3\sigma) \approx 0.997$
  - Note that these are very similar to Chebyshev's inequality, and it is a good exercise to show that these are indeed consistent with Chebyshev.
- Be able to read a standard normal table.
- Standard normal: $Z \sim \mathrm{N}(0, 1)$.
- If $X \sim \mathrm{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim \mathrm{N}(0, 1)$.
- Standard normal cumulative probabilities: $\mathrm{P}(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-z^2/2} dz$
- If $X \sim \mathrm{N}(\mu, \sigma^2)$, then $\mathrm{P}(X \leq x) = \mathrm{P}\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) = \mathrm{P}\left(Z \leq \frac{x-\mu}{\sigma}\right)$ which is now a standard normal cumulative probability.

Example problems:

1. If X is normally distributed with mean 50 and standard deviation 2, estimate $\mathrm{P}(X < 46)$.

   Solution:

   This is the probability that X is 2σ below μ or less. We can use the 68-95-99.7 rule and symmetry:

   $P(X < \mu - 2\sigma) = \frac{1}{2}(1 - \mathrm{P}(|x-\mu| \leq 2\sigma)) \approx \frac{1}{2}(1 - 0.95) = 0.025$

2. Identify the probability distribution for random variable X whose moment generating function is $M_X(t) = e^{5t+t^2}$.

   Solution:

   As we discussed in class on a few occasions, the moment gnerating function and probability distribution are unique to each random variable. So if we are given a probability distribution, then we know everything about the random variable. Likewise, if we are given the moment generating function, then we can determine what the probability distribution is.

The mgf we are given in this problem resembles $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ for $X \sim \mathrm{N}(\mu, \sigma^2)$

Thus we know our random variable is normally distributed with mean $\mu = 5$ and variance $\sigma^2 = 2$.

3. For $X \sim \mathrm{N}(\mu = 100, \sigma^2 = 25)$, find $\mathrm{E}[(X-100)^n]$ for $n = 1, 2, 3, 4, 5, 6, 7, 8$.

   Solution:

   These are the central moments. They can be calculated from integrals by symmetry arguments and integration by parts. The odd central moments are zero since a substitution in the integral of $z = \frac{x-\mu}{\sigma}$ would result in integrating an odd function. The even central moments are given by the formula $\mathrm{E}[(X-\mu)^n] = (n-1)!!\sigma^n$.

   $\mathrm{E}[(X-100)^2] = \sigma^2$

   $\mathrm{E}[(X-100)^4] = 3\sigma^4$

   $\mathrm{E}[(X-100)^6] = 5 \cdot 3 \cdot \sigma^6 = 15\sigma^6$

   $\mathrm{E}[(X-100)^8] = 7 \cdot 5 \cdot 3 \cdot \sigma^8 = 105\sigma^8$