Chapter 4 Summary and Review (draft: 2019/11/17-21:15:28)

Expected value

Summary of topics and terminology:

- Expected value is the same as what we normally think of as 'average' or 'mean'.
- $\mathrm{E}(X) = \int_{-\infty}^{\infty} x f_X(x) dx$ for continuous real-valued random variable X with pdf $f_X(x)$.
- $\mathrm{E}(X) = \sum_x x f_X(x)$ for discrete random variable X with pmf $f_X(x)$.
- Expected value of a function of RV X: $\mathrm{E}(u(X)) = \int_{-\infty}^{\infty} u(x) f_X(x) dx$ or $\sum_x u(x) f_X(x)$
- Expectation E() is a 'linear operator'. That means it distributes over sums and differences nicely, and that we can pull out constant coefficients:
  - $\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$
  - $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$
  - $\mathrm{E}(\sum_k X_k) = \sum_k \mathrm{E}(X_k)$
  - $\mathrm{E}(\sum_k c_k u_k(X)) = \sum_k c_k \mathrm{E}(u_k(X))$
- Multivariate expectation: $\mathrm{E}(u(X, Y)) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y) f_{X,Y}(x, y)\ dx\ dy$ or $\sum_x \sum_y u(x, y) f_{X,Y}(x, y)$

Example problems:

1. Givene pdf $f(x) = 2x$ for $0 < x < 1$. Find $\mathrm{E}(X)$.

   Solution:

   $\mathrm{E}(X) = \int_0^1 x \cdot 2x\ dx = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}$.

2. Consider a bag with 4 red balls and 3 blue balls. Draw 5 w/o replacement. Let X be the number of blue balls drawn. Find $\mathrm{E}(X)$.

   Solution:

   $P(X = x) = \dfrac{\binom{4}{5-x}\binom{3}{x}}{\binom{7}{5}}$ for $x = 1, 2, 3$.

   $\mathrm{E}(X) = 1 \cdot \dfrac{\binom{4}{4}\binom{3}{1}}{\binom{7}{5}} + 2 \cdot \dfrac{\binom{4}{3}\binom{3}{2}}{\binom{7}{5}} + 3 \cdot \dfrac{\binom{4}{2}\binom{3}{3}}{\binom{7}{5}} = \dfrac{1}{7} + \dfrac{8}{7} + \dfrac{6}{7} = \dfrac{15}{7} \approx 2.14$

3. Consider the joint pdf $f(x, y) = 6e^{-2x-3y}$ for $x > 0$, $y > 0$. Calculate $\mathrm{E}(X + Y)$.

   Solution:

   $\mathrm{E}(X+Y) = \int_0^\infty \int_0^\infty (x+y)6e^{-2x-3y} dx dy = \int_0^\infty \int_0^\infty x2e^{-2x} \int_0^\infty 3e^{-3y} dy + \int_0^\infty \int_0^\infty 2e^{-2x} \int_0^\infty y3e^{-3y} dy = \frac{1}{2} + \frac{1}{3}$

   Also notice that the joint pdf is separable and $f(x, y) = 2e^{-2x}3e^{-3y}$ Thus X and Y are independent exponential random variables and $\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y) = \frac{1}{2} + \frac{1}{3}$.

Moments

Summary of topics and terminology:

- Moments can be thought of the average value of the RV raised to a power.
- n^{th} raw moment or moment about the origin: $m_n = \mu_n' = \mathrm{E}(X^n) = \int_{-\infty}^{\infty} x^n f_X(x)dx$ for continuous real-valued random variable X with pdf $f_X(x)$.
- $m_n = \mu_n' = \mathrm{E}(X^n) = \sum_x x^n f_X(x)$ for discrete random variable X with pmf $f_X(x)$.
- n^{th} central moment or moment about the mean: $\mu_n = \mathrm{E}((X-\mu)^n) = \int_{-\infty}^{\infty}(x-\mu)^n f_X(x)dx$ for continuous and $\mu_n = \mathrm{E}((X-\mu)^n) = \sum_x (x-\mu)^n f_X(x)$ for discrete
- Multivariate moments: $\mu_{n,k}' = \mathrm{E}(X^n Y^k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y)\ dx\ dy$ or $\sum_x \sum_y x^n y^k f_{X,Y}(x,y)$
- Variance is the 2^{nd} central moment: $\mathrm{Var}(X) = \sigma_X^2 = \mathrm{E}[(X-\mu)^2] = \mathrm{E}(X^2) - \mu_X^2$.
- Moment generating function: $M_X(t) = \mathrm{E}(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$ or $\sum_x e^{tx} f_X(x)$
- MGF is a power series with moments as coefficients: $M_X(t) = 1 + \mathrm{E}(X)t + \frac{1}{2}\mathrm{E}(X^2)t^2 + \frac{1}{3!}\mathrm{E}(X^3)t^3 + \cdots$
- Moments can be calculated from MGF by taking derivatives and evaluating at $t=0$: $\mathrm{E}(X^n) = M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t)\big|_{t=0}$.
- Chebyshef's theorem: $P(|X-\mu| > k\sigma^2) \leq \frac{1}{k^2}$.

Example problems:

1. Givene pdf $f(x) = 2x$ for $0 < x < 1$. Find $\mu, \mu_2', \mu_2, \sigma^2$.

   Solution:

   $\mu = \mathrm{E}(X) = \int_0^1 x \cdot 2x\ dx = \frac{2}{3}x^3\big|_0^1 = \frac{2}{3}$.

   $\mu_2' = \mathrm{E}(X^2) = \int_0^1 x^2 \cdot 2x\ dx = \frac{2}{4}x^4\big|_0^1 = \frac{1}{2}$.

   $\sigma^2 = \mu_2 = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = \mu_2' - \mu^2 = \frac{1}{2} - (2/3)^2 = \frac{1}{18}$.

   So this random variable has mean $\frac{2}{3}$ and variance $\mathrm{Var}(X) = \frac{1}{18}$.

2. Find the moment generating function for X with pdf $f(x) = 2x$ on $[0,1]$.

   Solution:

   $M(t) = \mathrm{E}(e^{tX}) = \int_0^1 e^{tx} 2x\ dx = 2x\frac{1}{t}e^{tx}\big|_0^1 - 2\int_0^1 \frac{1}{t}e^{tx}dx = 2\frac{1}{t}e^t - 2\ \frac{1}{t^2}e^{tx}\big|_0^1 = 2\left(\frac{1}{t}e^t - \frac{1}{t^2}e^t + \frac{1}{t^2}\right)$

3. Given moment generating function $M_X(t) = \frac{1}{1-t}e^{t^2}$ find $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$.

   Solution:

   $M'(t) = (1-t)^{-2}e^{t^2} + (1-t)^{-1}2te^{t^2}$ thus $M'(0) = 1 = \mathrm{E}(X)$.

   $M''(t) = 2(1-t)^{-3}e^{t^2} + (1-t)^{-2}2te^{t^2} + (1-t)^{-2}2te^{t^2} + (1-t)^{-1}2e^{t^2} + (1-t)^{-1}4t^2e^{t^2}$ thus $M''(0) = 4 = \mathrm{E}(X^2)$.

Linear combinations and covariance

Summary of topics and terminology:

- Linear combination of RVs $X_1, X_2, \ldots$ is $\sum_{i=1}^{\infty} c_i X_i$
- $\sigma_{XY} = \mathrm{Cov}(X,Y) = \mathrm{E}((X-\mu_X)(Y-\mu_Y)) = \mathrm{E}(XY) - \mu_X\mu_Y$ and can be positive or negative or zero.
- RVs X and Y independent implies that they have zero covariance, but zero covariance does not imply independence.
- $Y_1 = \sum_i a_i X_i$ and $Y_2 = \sum_i b_i X_i$, then:
  - $\mathrm{Var}(Y_1) = \sum_i a_i^2 \mathrm{Var}(Y_i) + 2\sum_{i<j} a_i a_j \mathrm{Cov}(X_i, X_j)$
  - $\mathrm{Cov}(Y_1, Y_2) = \sum_i a_i b_i \mathrm{Var}(X_i) + \sum_{i<j}(a_i b_j + a_j b_i)\mathrm{Cov}(X_i, X_j)$

Example problems:

1. Let $Y_1 = 3X_1 - X_2$ and $Y_2 = X_1 + 5X_2$. Given that X_1 and X_2 have variances, 1 and 2, respectively, and covariance -1, find the covariance of Y_1 and Y_2.

   Solution:

   $\mathrm{Cov}(Y_1, Y_2) = 3 \cdot 1 \cdot \mathrm{Var}(X_1) + (-1) \cdot 5 \cdot \mathrm{Var}(X_2) + (3 \cdot 5 + (-1) \cdot 1)\mathrm{Cov}(X_1, X_2)$
   $= 3 \cdot 1 - 5 \cdot 2 + 14 \cdot (-1) = -21$

2. Given joint pdf $f(x,y) = 2$ on $0 < x < y < 1$, find $\mathrm{Cov}(X,Y)$.

   Solution:

   $\mu_X = \int_{-\infty}^{\infty} x f(x,y)\, dx\, dy = \int_0^1 \int_0^y 2x\, dx\, dy = \frac{1}{3}$

   $\mu_Y = \int_{-\infty}^{\infty} y f(x,y)\, dx\, dy = \int_0^1 \int_0^y 2y\, dx\, dy = \frac{2}{3}$

   $\mathrm{E}(XY) = \int_{-\infty}^{\infty} xy f(x,y)\, dx\, dy = \int_0^1 \int_0^y 2xy\, dx\, dy = \frac{1}{4}$

   $\sigma_{XY} = \mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mu_X\mu_Y = \frac{1}{4} - \frac{1}{3} \cdot \frac{2}{3} = \frac{1}{36}$

3. If $\mathrm{Cov}(X,Y) = 0$, are X and Y independent?

   Solution:

   No! Independent RVs do have zero covariance, but zero covariance does not imply independence!

Conditional expectation

Summary of topics and terminology:

- $\mathrm{E}(X \mid X \in A) = \dfrac{\int_A x f_X(x)\ dx}{P(X \in A)}$
- $\mathrm{E}(X \mid Y \in A) = \dfrac{\int_{-\infty}^{\infty} \int_A x f(x,y)\ dy\ dx}{P(Y \in A)}$
- $\mathrm{E}(X \mid Y = y) = \displaystyle\int_{-\infty}^{\infty} x f_{X|Y=y}(x|y)\ dx = \int_{-\infty}^{\infty} x \frac{f(x,y)}{f_Y(y)}\ dx$ for y a *fixed* value
- The above formulas are given for continuous RVs, but the formulas for discrete RVs would be the same but with summations instead of integrals.
- We can take the expectation of functions of RVs this way as well, e.g. $E(u(X) \mid X \in A) = \frac{\int_A u(x) f_X(x)\ dx}{P(X \in A)}$
- $E(u(X,Y) \mid X \in A, Y \in B) = \frac{\int_B \int_A u(x,y) f(x,y)\ dx\ dy}{P(X \in A, Y \in B)}$
- Conditional mean: $\mu_{X|y} = \mathrm{E}(X \mid Y = y)$
- Conditional variance: $\sigma^2_{X|y} = \mathrm{E}(X^2 \mid Y = y) - \mu^2_{X|y}$

Example problems:

1. Consider pmf $f(x) = x/6$ for $x = 1, 2, 3$. Calculate $\mathrm{E}(X \mid X \neq 3)$.

   Solution:

   $\mathrm{E}(X \mid X \neq 3) = \frac{1 \cdot \frac{1}{6} + 2 \cdot \frac{2}{6}}{\frac{1}{6} + \frac{2}{6}} = \frac{5}{3}$

2. Given pdf $f(x) = \frac{2}{x^3}$ for $x > 1$, find $\mathrm{E}(X \mid X > t)$.

   Solution:

   $$\mathrm{E}(X \mid X > t) = \frac{\int_t^{\infty} x \frac{2}{x^3} dx}{\int_t^{\infty} \frac{2}{x^3} dx} = \frac{-2x^{-1}|_t^{\infty}}{-x^{-2}|_t^{\infty}} = \frac{2/t}{1/t^2} = 2t$$

   This means that, given that we know X is beyond some threshold value, we expect it to be twice that threshold.

3. Consider joint pdf $f(x,y) = \frac{x}{y^2} + \frac{1}{2} e^{1-y}$ n $[0,1] \times [1, \infty)$. Calculate the expected value of X given that Y is less than 2. Also calculate the conditional expectation of X given that Y is exactly 2.

   Solution:

   $$\mathrm{E}(X \mid Y < 2) = \frac{\int_0^1 \int_1^2 x \left(\frac{x}{y^2} + \frac{1}{2} e^{1-y} \right) dy\ dx}{\int_0^1 \int_1^2 \left(\frac{x}{y^2} + \frac{1}{2} e^{1-y} \right) dy\ dx}$$

   $$= \frac{\frac{1}{6} + \frac{e-1}{4e}}{\frac{1}{4} + \frac{e-1}{2e}} = \frac{4e + 6(e-1)}{6e + 12(e-1)} = \frac{5e-3}{9e-6} \approx 0.5736$$

   The marginal for Y is $f_Y(y) = \frac{1}{2y^2} + \frac{1}{2} e^{1-y}$.

   $$\mathrm{E}(X \mid Y = 2) = \int_0^1 x \frac{f(x,2)}{f_Y(2)}\ dx = \int_0^1 x \frac{\frac{x}{4} + \frac{1}{2} e^{-1}}{\frac{1}{8} + \frac{1}{2} e^{-1}}\ dx = \frac{1}{1 + 4e^{-1}} \int_0^1 2x^2 + 4xe^{-1}\ dx$$

   $$= \frac{1}{1 + 4e^{-1}} \left[\frac{2}{3} x^3 + 2x^2 e^{-1} \right]_0^1 = \frac{\frac{2}{3} + 2e^{-1}}{1 + 4e^{-1}} \approx 0.5674$$