A Characterization of Uniquely Pressable Simple Pseudo-Graphs

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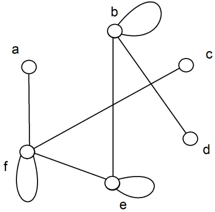
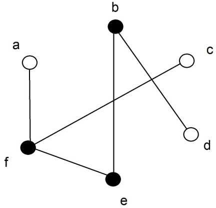
Simple Pseudo-Graph





The Pressing Game

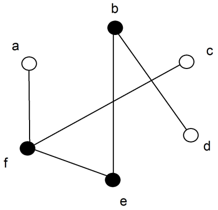
Definition 1

Let G be a simple-pseudo graph with a looped vertex $v \in V(G)$. "Pressing v" is the operation of transforming G into $G(v)$, a new loopy graph in which $G[N(v)]$ is complemented. That is,

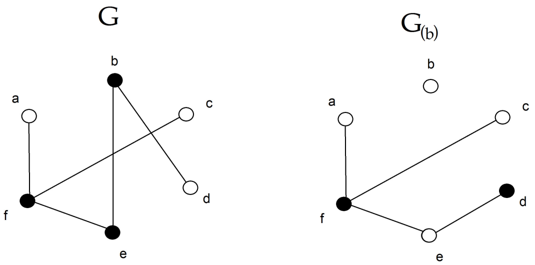
$$V(G(v)) = V(G), \quad E(G(v)) = E(G)\Delta\{N(v) \times N(v)\}$$

The Pressing Game

G



THE PRESSING GAME



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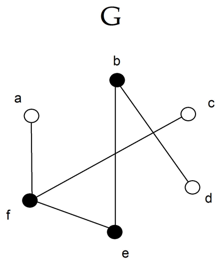

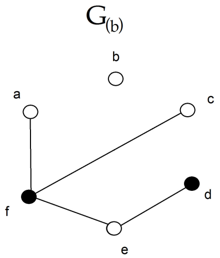

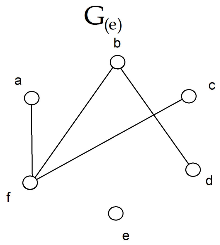

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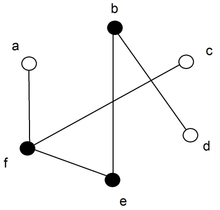
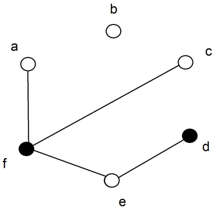
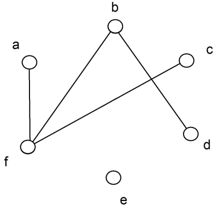


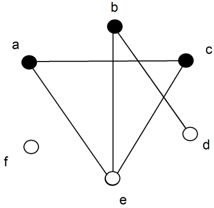
$G_{(b)}$



$G_{(e)}$



$G_{(f)}$



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- We say $v_1, v_2, ..., v_k$ is a successful pressing sequence for G if pressing the vertices $v_1, v_2, ..., v_k$ in order transform G into the edgeless graph:

$$G_{(v_1, v_2, \ldots, v_k)} = (V(G), \emptyset)$$

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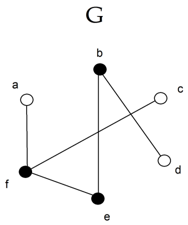

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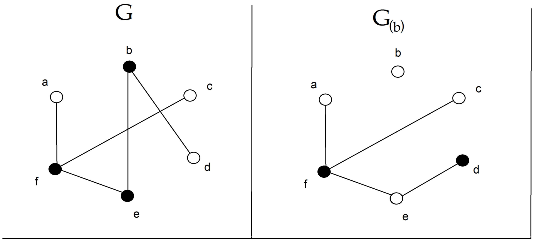

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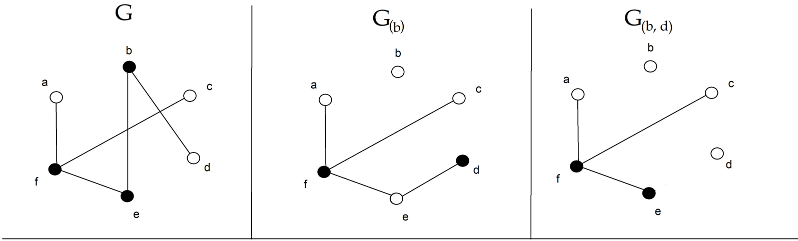

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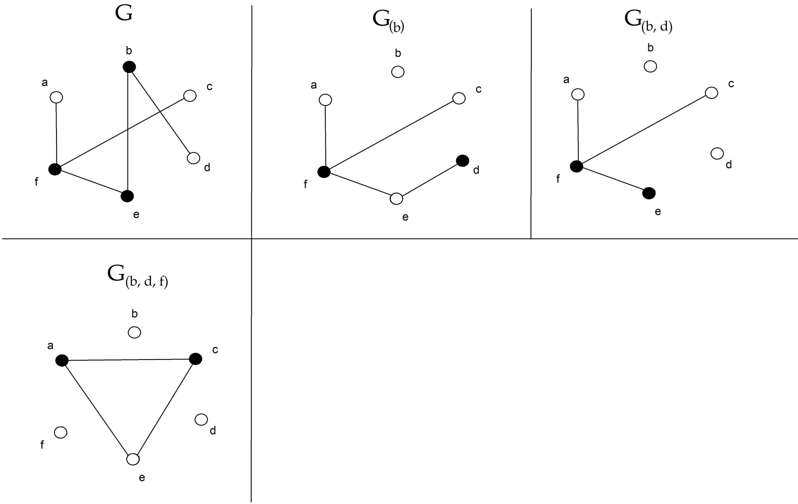

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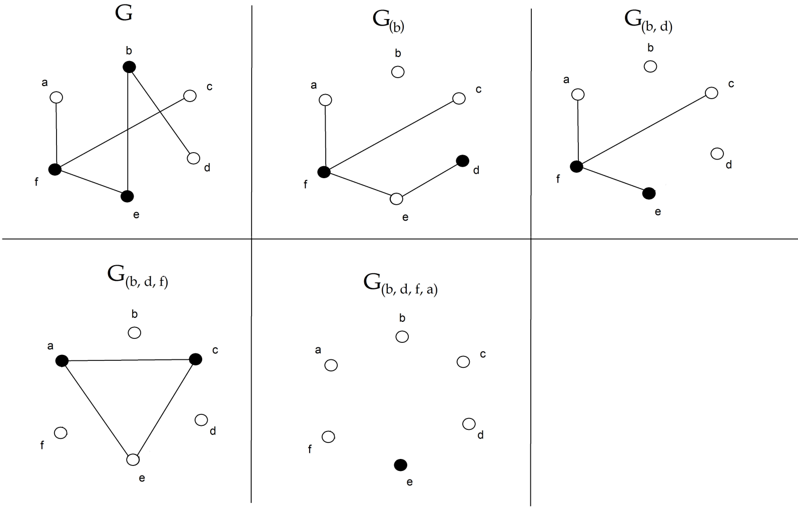

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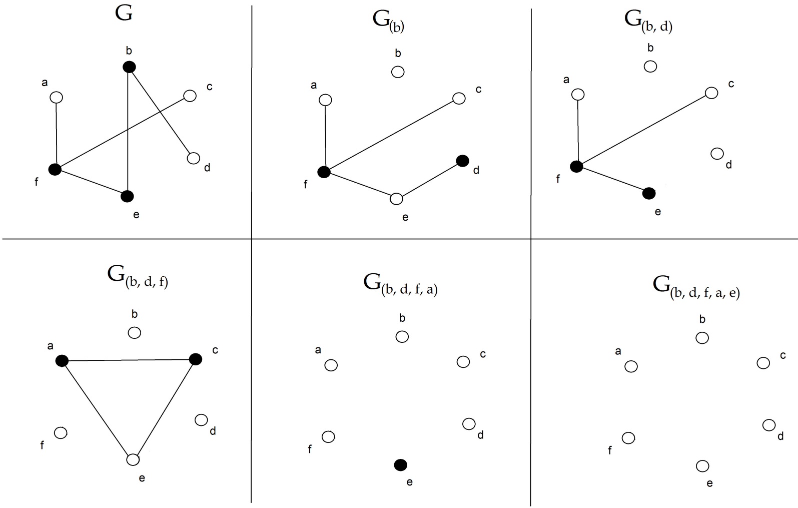

Some Notation

- An ordered, simple pseudo-graph, abbreviated OSP-graph, is a simple pseudo-graph with a total order over its vertices.

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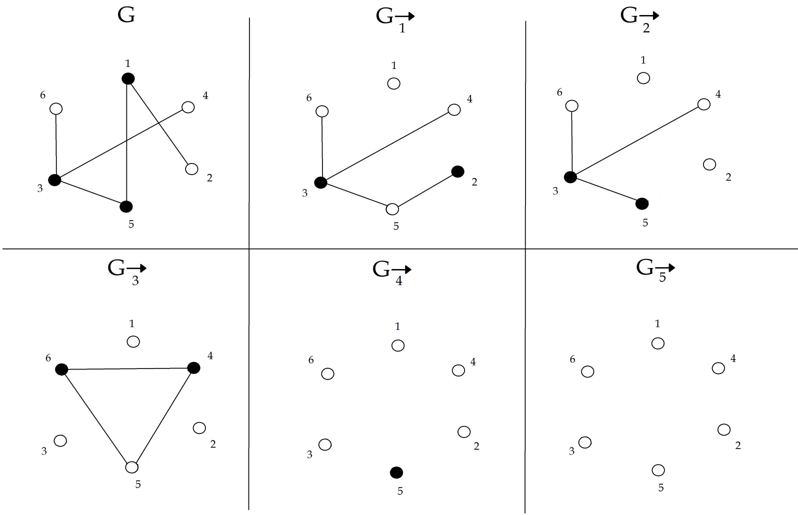
- An OSP-graph G is said to be order-pressable if there exists some initial segment of $V(G)$ that is a successful pressing sequence.
- If $G = ([n], E)$ is order-pressable then we let $G_{\vec{k}}$ denote the result of pressing vertices $1, 2, \ldots, k$ in order.


G
1
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$G_{\vec{1}}$
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Proof:

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Proof:

- wlog $V(G) = [n]$
- If pressing length is $m < n$ then $G, G_{\vec{1}}, G_{\vec{2}}, ..., G_{\vec{m}} = ([n], \emptyset)$

Lemma 3

If G is a connected OSP-graph that is uniquely pressable then the pressing length of G is $|V(G)|$.

Proof:

$$G, G_{\vec{1}}, G_{\vec{2}}, ..., G_{\vec{m}} = ([n], \emptyset)$$

- Let $k = \min_{i \in [m]} \{ i \mid G_{\vec{i}} \text{ has more than } i \text{ isolated vertices} \}$.

Lemma 4

If G is a connected OSP-graph that is uniquely pressable then the pressing length of G is $|V(G)|$.

Proof:

$$G, G_{\vec{1}}, G_{\vec{2}}, ..., G_{\overrightarrow{k-1}}, , G_{\vec{k}}, ..., G_{\vec{m}} = ([n], \emptyset)$$

- There exist some vertex $\ell > k$ such that

$$N_{G_{\overrightarrow{k-1}}}(\ell) = N_{G_{\overrightarrow{k-1}}}(k)$$

Lemma 5

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$$N_{G_{\overrightarrow{k-1}}}(\ell) = N_{G_{\overrightarrow{k-1}}}(k)$$

- $1, ..., k-1, \ell, k+1, ..., \ell-1, k, \ell+1, ..., m$ is also a successful pressing sequence for G.

LEMMA 6 (COOPER/DAVIS)

A simple pseudo-graph G contains a successful pressing sequence if and only if every non-trivial component of G contains a looped vertex.

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Corollary 7

If G is a uniquely pressable OSP-graph with at least one edge then G contains exactly one non-trivial component C and the pressing length of G is $|V(C)|$.

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In order to understand uniquely pressable OSP-graphs it will suffice to understand connected, uniquely-pressable OSP-graphs.

CUP$_n$ is the set of connected, uniquely pressable, ordered ($<_{\mathbb{N}}$), simple pseudo-graphs on n positive-integer vertices.

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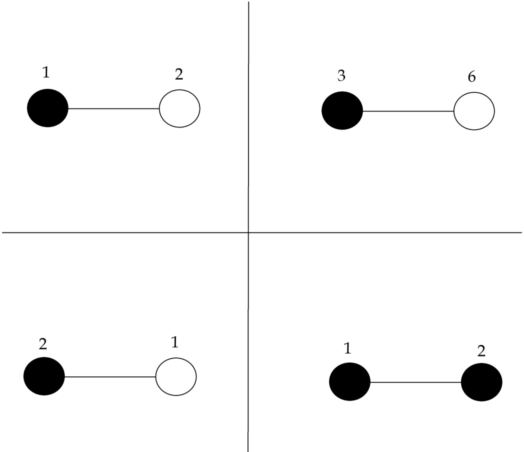

Definition 8

Given an OSP-graph $G = ([n], E)$ define adjacency matrix $A = A(G) = (a_{i,j}) \in \mathbb{F}_2^{n\times n}$ by

$$a_{i,j} = \begin{cases} 1 & \text{if } ij \in E, \\ 0 & \text{otherwise.} \end{cases}.$$

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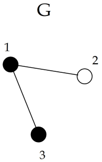

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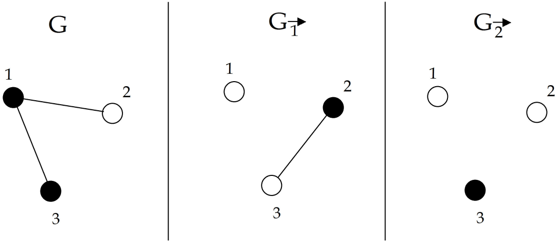

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Definition 9

The instructional Cholesky root $U = U(G)$ is defined by $U = (u_{i,j}) \in \mathbb{F}_2^{n \times n}$ by

$$u_{i,j} = \begin{cases} 1 & \text{if } i \leq j \text{ and } j \in N_{G_{\overrightarrow{i-1}}}(i), \\ 0 & \text{otherwise.} \end{cases}.$$

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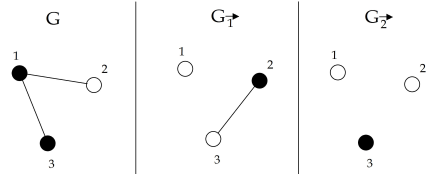

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$u_{i,j} = \begin{cases} 1 & \text{if } i \leq j \text{ and } j \in N_{\overrightarrow{G_{i-1}}}, \\ 0 & \text{otherwise.} \end{cases}.$$

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U

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$$\underset{U^T}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

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The instructional Cholesky root U of G is a Cholesky root of the adjacency matrix A of G.

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$$S_{i,j} = \{t \in [k] \colon ij \in E(G_{\overrightarrow{t-1}}) \triangle E(G_{\overrightarrow{t}})\}$$

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- $a_{i,j} \equiv |S_{i,j}|$
- $S_{i,j} = T_{i,j}$
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$$b_{i,j} \equiv |T_{i,j}| = |S_{i,j}| \equiv a_{i,j}$$

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Given $M \in \mathbb{F}_2^{n \times n}$ with rows and columns indexed identically by $[n]$:

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Given $M \in \mathbb{F}_2^{n \times n}$ with rows and columns indexed identically by $[n]$:
$M_{\hat{i}}$ is the result of removing row and column i from M.

Lemma 10

If $G \in \mathbf{CUP}_n$ then $G^{\overrightarrow{1}} \in \mathbf{CUP}_{n-1}$ and the instructional Cholesky root of $G^{\overrightarrow{1}}$ is $U_{\hat{1}}$.

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The unique pressing sequence of $G_{\overrightarrow{1}}$ is realized by

$$G_{\overrightarrow{1}}, G_{\overrightarrow{2}}, ..., G_{\overrightarrow{n}}$$

and so the unique pressing sequence of $G^{\overrightarrow{1}}$ is realized by

$$G_{\overrightarrow{1}} - 1, G_{\overrightarrow{2}} - 1, ..., G_{\overrightarrow{n}} - 1$$

Lemma 11

Let $G \in \mathbf{CUP}_n$ and let $H = G - n$ be the induced subgraph of G on $[n-1]$. Then $H \in \mathbf{CUP}_{n-1}$ and the instructional Cholesky root of H is $U_{\hat{n}}$.

$H = G - n \in \mathbf{CUP}_{n-1}$

Proof Outline:

$H = G - n \in \mathbf{CUP}_{n-1}$

Proof Outline: For all $j \in [n-1]$:

$$
\begin{aligned}
N_{H_{\vec{1}}}(j) &= \begin{cases} N_H(j) \triangle N_H(1), & 1j \in E(H) \\ N_H(j), & 1j \notin E(H) \end{cases} \\
&= \begin{cases} \{N_G(j) \triangle N_G(1)\} - \{n\}, & 1j \in E(H) \leftrightarrow 1j \in E(G) \\ N_G(j) - \{n\}, & 1j \notin E(H) \leftrightarrow 1j \notin E(G) \end{cases} \\
&= \begin{cases} N_{G_{\vec{1}}}(j) - \{n\}, & 1j \in E(G) \\ N_{G_{\vec{1}}}(j) - \{n\}, & 1j \notin E(G) \end{cases}
\end{aligned}
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&= \begin{cases} N_{G_{\vec{1}}}(j) - \{n\}, & 1j \in E(G) \\ N_{G_{\vec{1}}}(j) - \{n\}, & 1j \notin E(G) \end{cases}
\end{aligned}$$

1 is a looped vertex and $H_{\vec{1}} = G_{\vec{1}} - n$

$H = G - n \in \mathbf{CUP}_{n-1}$

Proof Outline: For all $j \in [n-1]$:

$$
\begin{aligned}
N_{H_{\overrightarrow{i+1}}}(j) &= \begin{cases} N_{H_{\vec{i}}}(j) \triangle N_{H_{\vec{i}}}(1), & 1j \in E(H_{\vec{i}}) \\ N_{H_{\vec{i}}}(j), & 1j \notin E(H_{\vec{i}}) \end{cases} \\
&= \begin{cases} \left\{ N_{G_{\vec{i}}}(j) \triangle N_{G_{\vec{i}}}(1) \right\} - \{n\}, & 1j \in E(G_{\vec{i}}) \\ N_{G_{\vec{i}}}(j) - \{n\}, & 1j \notin E(G_{\vec{i}}) \end{cases} \\
&= \begin{cases} N_{G_{\overrightarrow{i+1}}}(j) - \{n\}, & 1j \in E(G_{\vec{i}}) \\ N_{G_{\overrightarrow{i+1}}}(j) - \{n\}, & 1j \notin E(G_{\vec{i}}) \end{cases}
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&= \begin{cases} N_{G_{\overrightarrow{i+1}}}(j) - \{n\}, & 1j \in E(G_{\vec{i}}) \\ N_{G_{\overrightarrow{i+1}}}(j) - \{n\}, & 1j \notin E(G_{\vec{i}}) \end{cases}
\end{aligned}
$$

$1, 2, \ldots, n-1$ is a valid pressing sequence for H that is encoded by $U_{\hat{n}}$.

$H = G - n \in \mathbf{CUP}_{n-1}$

H is order pressable so $adj(H) = V^T V$ for some instructional C. We need to show that H is uniquely pressable.

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Let $\sigma = (v_1, v_2, ..., v_{n-1})$ be a valid pressing sequence for H and let $\tau = (v_1, v_2, ..., v_{n-1}, n)$.

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$$\det(PAP^T) = \det(P)\det(A)\det(P^T) =$$

$$= \det(A) = \det(U^T)\det(U) = 1 \neq 0$$

Since $A = A(G)$ then $adj(H) = A_{\hat{n}}$ and

$$PAP^T = \left[\begin{array}{c|c} P_{\hat{n}} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right] \left[\begin{array}{c|c} A_{\hat{n}} & * \\ \hline * & * \end{array}\right] \left[\begin{array}{c|c} P_{\hat{n}} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right]^T$$

$$= \left[\begin{array}{c|c} P_{\hat{n}} A_{\hat{n}} P_{\hat{n}}{}^T & * \\ \hline * & * \end{array}\right] = \left[\begin{array}{c|c} V^T V & * \\ \hline * & * \end{array}\right]$$

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So every leading principal minor of PAP^T is nonsingular

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$$= \left[\begin{array}{c|c} P_{\hat{n}} A_{\hat{n}} P_{\hat{n}}{}^T & * \\ \hline * & * \end{array}\right] = \left[\begin{array}{c|c} V^T V & * \\ \hline * & * \end{array}\right]$$

So every leading principal minor of PAP^T is nonsingular so τ is a valid pressing sequence for G.

$$\tau = \overrightarrow{n} \quad \text{and hence } \sigma = \overrightarrow{n-1}$$

Corollary 12

Let $G \in \mathbf{CUP}_n$ with instructional Cholesky root U. Then any principal submatrix of U on k consecutive rows and columns is the instructional Cholesky root of a $\mathbf{CUP}_k$ graph.

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Corollary 13

If U is the instructional Cholesky root of $G \in \mathbf{CUP}_n$ then U must have all 1's on the main diagonal and super-diagonal.

Definition 14

Given an OSP-graph $G = ([n], E)$ with instructional Cholesky root U we define vertex weight by

$$wt(j) = \sum_{i \in [n]} u_{i,j}$$

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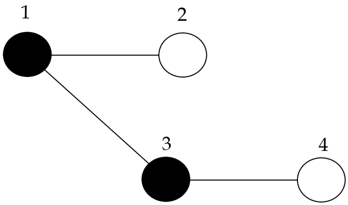


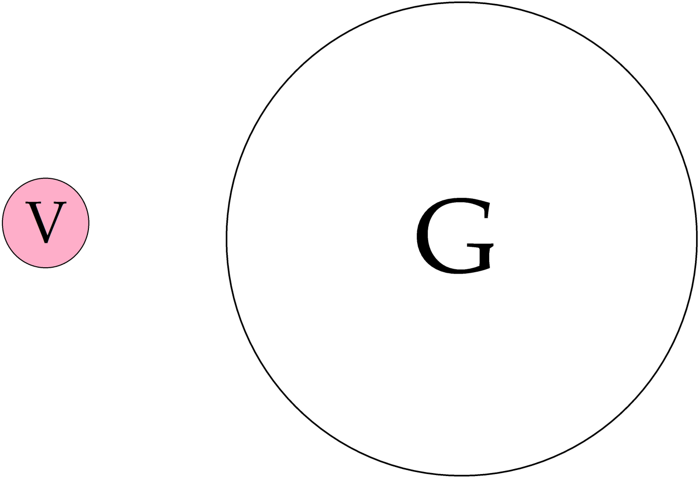
$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

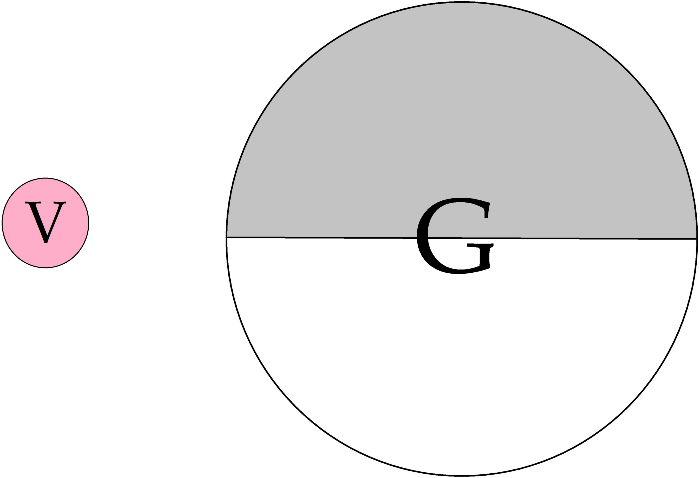
$$U = \left[\begin{array}{c|c} * & * \\ \hline * & U_{\hat{1}} \end{array}\right] = \left[\begin{array}{c|c} U_{\hat{n}} & * \\ \hline * & * \end{array}\right]$$

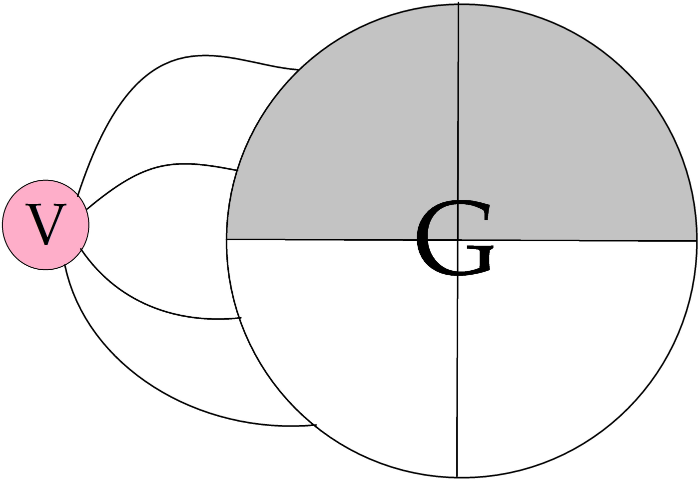
$$U = \left[\begin{array}{c|c} 1 & * \\ \hline 0 & \\ \vdots & U_{\hat{1}} \\ 0 & \end{array}\right] = \left[\begin{array}{c|c} U_{\hat{n}} & * \\ \hline 0 \cdots 0 & 1 \end{array}\right]$$

$$A = U^T U \quad = \quad \left[\begin{array}{c|c} 1 & * \\ \hline * & U_{\hat{1}}^{\;T} U_{\hat{1}} \end{array}\right]$$

$$= \quad \left[\begin{array}{c|c} U_{\hat{n}}^T U_{\hat{n}} & * \\ \hline * & * \end{array}\right]$$


V
G


V
G


V
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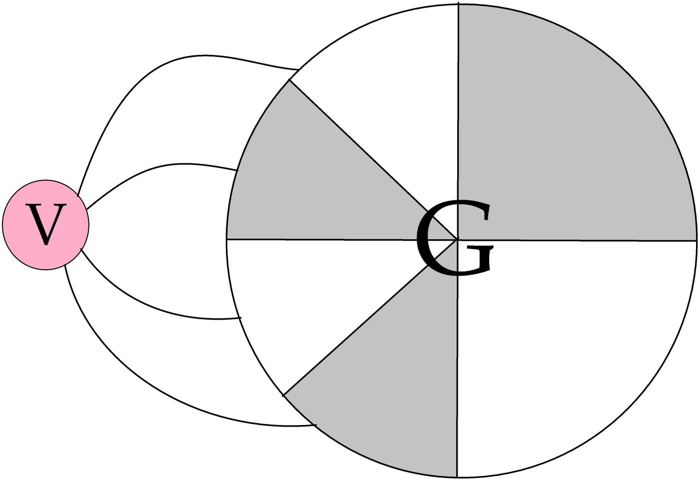
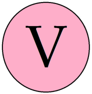
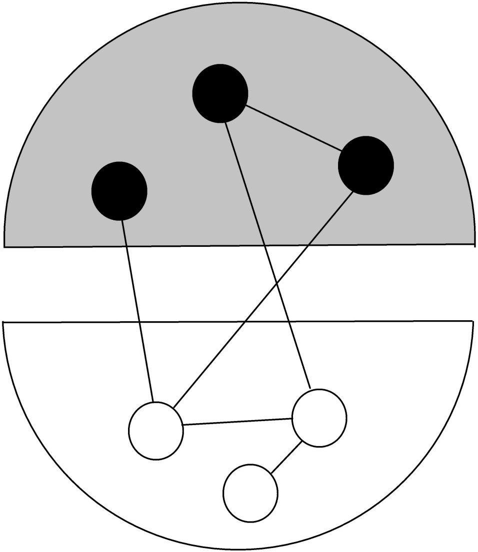
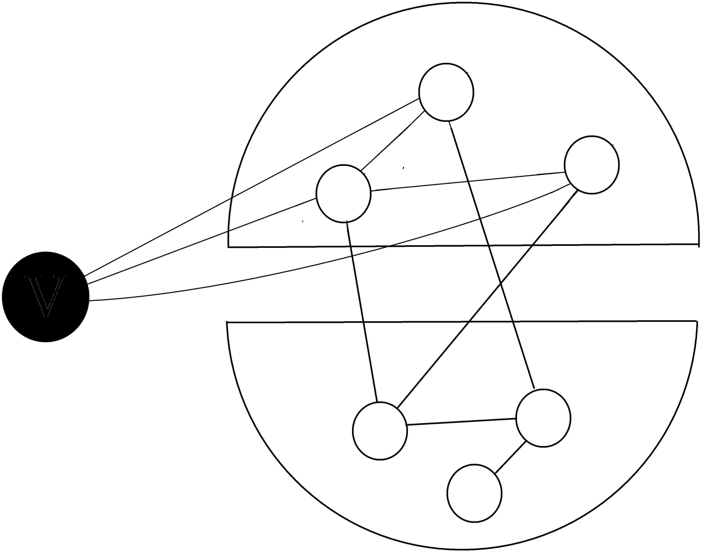

V
G

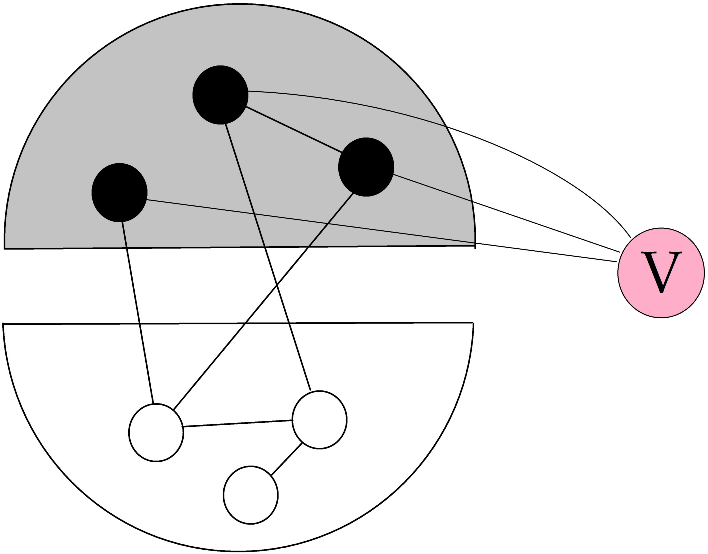

V




V

$$U = \left[\begin{array}{c|c} 1 & * \\ \hline 0 & \\ \vdots & U_{\hat{1}} \\ 0 & \end{array}\right]$$

where $*$ has 1's above the odd-weight columns of $U_{\hat{1}}$

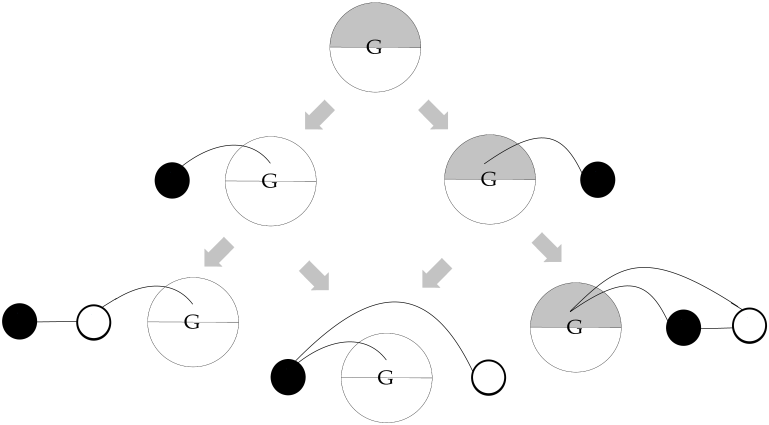

V

$$U = \left[\begin{array}{c|c} U_{\hat{n}} & \mathbf{1} \\ \hline 0 \cdots 0 & 1 \end{array}\right]$$

To show uniqueness of pressing sequence we show that

$$UP^T = QU' \quad \Leftrightarrow \quad P = Q = I$$

by Gram-Schmidt Orthonormalization.


G
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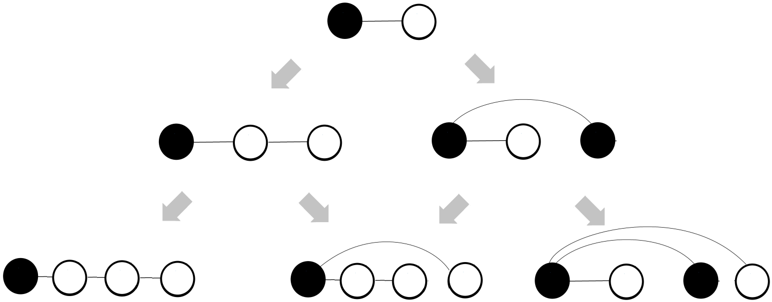
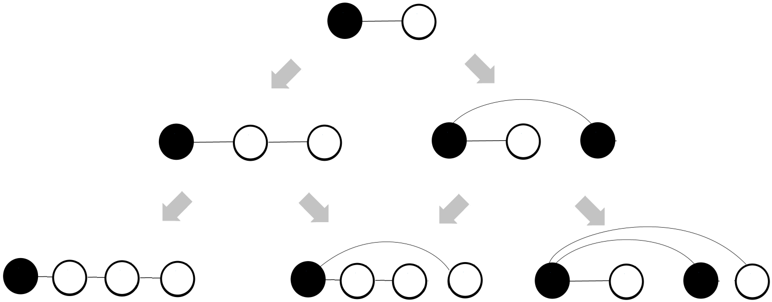


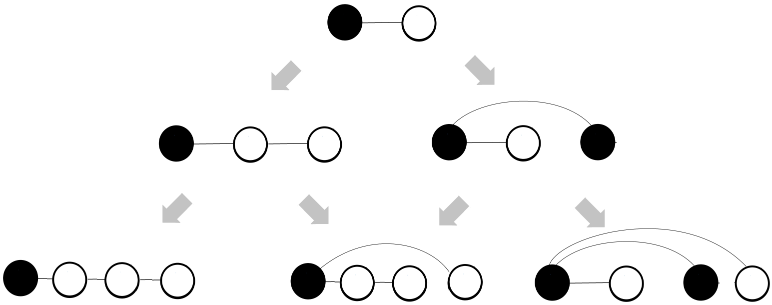


Number of connected uniquely pressable simple pseudo-graphs on vertex set $[n]$ is

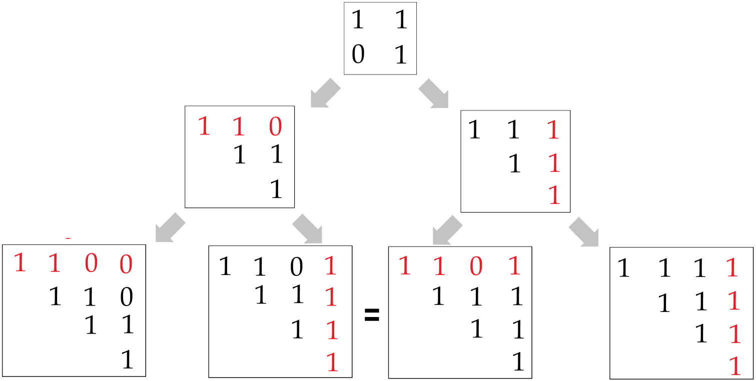
$$\geq \begin{cases} 3^{\frac{n}{2}-1} & n \text{ is even} \\ 2 \cdot 3^{\frac{n-1}{2}-1} & n \text{ is odd} \end{cases}$$

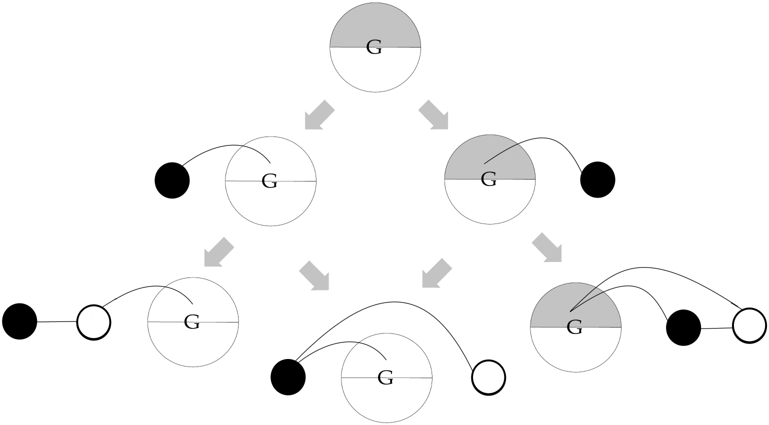
Number of ~~connected~~ uniquely pressable simple pseudo-graphs on vertex set $[n]$ is

$$\geq \begin{cases} \frac{1}{2}\left(5\left(\sqrt{3}\right)^{n-2}+1\right) & n \text{ is even} \\ \frac{1}{2}\left(\left(\sqrt{3}\right)^{n+1}+1\right) & n \text{ is odd} \end{cases}$$



A Characterization of Uniquely Pressable Simple Pseudo-Graphs




G
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G

Let $\mathcal{M}$ be the set of upper-triangular matrices U over $\mathbb{F}_2$ satisfying for all $i < j \in [n]$:

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- $u_{i,j} = 1 \Rightarrow u_{i+1,j} = 1$

Let $\mathcal{M}$ be the set of upper-triangular matrices U over $\mathbb{F}_2$ satisfying for all $i < j \in [n]$:

- $u_{i,i} = 1$
- $u_{i,j} = 1 \Rightarrow u_{i+1,j} = 1$
- $wt(i) \leq wt(i+1)$

Let $\mathcal{M}$ be the set of upper-triangular matrices U over $\mathbb{F}_2$ satisfying for all $i < j \in [n]$:

- $u_{i,i} = 1$
- $u_{i,j} = 1 \Rightarrow u_{i+1,j} = 1$
- $wt(i) \leq wt(i+1)$
- If $wt(i) > 2$ then $wt(i) < wt(i+2)$

Let $\mathcal{M}$ be the set of upper-triangular matrices U over $\mathbb{F}_2$ satisfying for all $i < j \in [n]$:

- $u_{i,i} = 1$
- $u_{i,j} = 1 \Rightarrow u_{i+1,j} = 1$
- $wt(i) \leq wt(i+1)$
- If $wt(i) > 2$ then $wt(i) < wt(i+2)$
- If $wt(i) \equiv 1$ then $wt(i) = i$ and if $i \neq 1$ then $wt(j) = j$ for all $j > i$

$$\mathcal{M} = \left\{ \begin{array}{ll} u_{i,i} = 1, & i \in [n] \\ u_{i,j} = 1 \Rightarrow u_{i+1,j} = 1, & i < j \in n \\ wt(i) \leq wt(i+1), & i \in [n-1] \\ wt(i) > 2 \Rightarrow wt(i) < wt(i+2), & i \in [n-2] \\ wt(i) \equiv 1 \Rightarrow wt(k) = k, & 1 < i \leq k \leq n \end{array} \right\}$$

$$\mathcal{M} = \left\{ \begin{array}{ll} u_{i,i} = 1, & i \in [n] \\ u_{i,j} = 1 \Rightarrow u_{i+1,j} = 1, & i < j \in n \\ wt(i) \leq wt(i+1), & i \in [n-1] \\ wt(i) > 2 \Rightarrow wt(i) < wt(i+2), & i \in [n-2] \\ wt(i) \equiv 1 \Rightarrow wt(k) = k, & 1 < i \leq k \leq n \end{array} \right\}$$

is the set of instructional Cholesky roots for the previously discussed uniquely pressables.

Theorem 15

If $G = ([n], E) \in \mathbf{CUP}_n$ then its instructional Cholesky root $U \in \mathcal{M}$.

Proof Outline:

$$U = \left[\begin{array}{c|c} 1 & * \\ \hline 0 & \\ \vdots & U_{\hat{1}} \\ 0 & \end{array}\right] = \left[\begin{array}{c|c} U_{\hat{n}} & * \\ \hline 0 \cdots 0 & 1 \end{array}\right]$$

Proof Outline:

$$U = \left[\begin{array}{c|c} 1 & * \\ \hline 0 & \\ \vdots & U_{\hat{1}} \\ 0 & \end{array}\right] = \left[\begin{array}{c|c} U_{\hat{n}} & * \\ \hline 0 \cdots 0 & 1 \end{array}\right]$$

The properties of $\mathcal{M}$ hold on $U_{\hat{1}}$ and $U_{\hat{n}}$ by inductive argument.

Proof Outline:

$$U = \left[\begin{array}{c|c} 1 & * \\ \hline 0 & \\ \vdots & U_{\hat{1}} \\ 0 & \end{array}\right] = \left[\begin{array}{c|c} U_{\hat{n}} & * \\ \hline 0 \cdots 0 & 1 \end{array}\right]$$

The properties of $\mathcal{M}$ hold on $U_{\hat{1}}$ and $U_{\hat{n}}$ by inductive argument. Hence the hold on the first $n-1$ columns of U.

Proof Outline:

$$U = \left[\begin{array}{c|c} 1 & * \\ \hline 0 & \\ \vdots & U_{\hat{1}} \\ 0 & \end{array}\right] = \left[\begin{array}{c|c} U_{\hat{n}} & * \\ \hline 0 \cdots 0 & 1 \end{array}\right]$$

The properties of $\mathcal{M}$ hold on $U_{\hat{1}}$ and $U_{\hat{n}}$ by inductive argument. Hence the hold on the first $n-1$ columns of U. We show they hold on the n^{th} column by case analysis of $u_{1,n}$ and $u_{2,n}$.

Theorem 16

$\mathcal{M}$ is the entire set of instructional Cholesky roots of uniquely pressable simple pseudo-graphs on $[n]$.

Corollary 17

The number of non-isomorphic uniquely pressable simple pseudo-graphs on n vertices is

$$\begin{cases} \frac{1}{2}\left(5\left(\sqrt{3}\right)^{n-2}+1\right) & n \text{ is even} \\ \frac{1}{2}\left(\left(\sqrt{3}\right)^{n+1}+1\right) & n \text{ is odd} \end{cases}$$

FURTHER WORK

- Do the linear extensions of the instructional Cholesky DAG always correspond to pressing sequences?

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- Is the pressing game conjecture true?

Further work

- Do the linear extensions of the instructional Cholesky DAG always correspond to pressing sequences?
- Is the pressing game conjecture true?
- Describe the pressing sequence meta-graph for $OSP - G(n, p)$ graphs.