Chapter 7 Summary and Review (draft: 2019/12/07-19:09:47)

Functions of random variable

Summary of topics and terminology:

- For random variable Y is a function os X, $Y = u(X)$ the pdf of Y is given by $f_Y(y) = f_X(u^{-1}(y)) \left|\frac{d}{dy}u^{-1}(y)\right|$
  Note that this requires the function $u(x)$ to be one-to-one.
- Cumulative probability technique: if $Y = u(X)$, then $F_Y(y) = P(Y \leq y) = P(u(X) \leq y)$. Then depending on the nature of $u(x)$, one can either find u^{-1} or otherwise find which intervals give this probability for X.
  $P(u(X) \leq y) = P(X \leq u^{-1}(y))$
- If $u(x)$ is not one-to-one, then you must carefully think about how "$u(X) \leq y$" converts to one or more intervals in the form $a(y) \leq X \leq b(y)$,
  e.g. $F_Y(y) = P(Y \leq y) = P(u(X) \leq y) = P(a(y) \leq X \leq b(y)) = F_X(b(y)) - F_X(a(y))$
  Then $f_Y(y) = f_X(b(y))|b'(y)| - f_X(a(y))|a'(y)|$.
- For an sum of independent random variables $Y = \sum_{i=1}^n X_i$ the mgf of Y is the product of the mgf's:
  $M_Y(t) = \prod_{i=1}^n M_{X_i}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)$
- If the X_i are i.i.d. with mgf $M_X(t)$ then $M_Y(t) = [M_X(t)]^n$

Example problems:

1. If X is exponentially distributed with mean θ, find the distribution for $Y = X^2$.

   Solution:

   $f_X(x) = \frac{1}{\theta}e^{-x/\theta}$ and $u(x) = x^2$ thus $u^{-1}(y) = \sqrt{y}$.

   So $f_Y(y) = f_X(\sqrt{y})\frac{1}{2\sqrt{y}} = \frac{1}{2\theta}y^{-1/2}e^{-\frac{1}{\theta}\sqrt{y}}$ for $y > 0$

2. Let X_i be i.i.d. and each uniformly distributed on $[0, 1]$. Find the moment generating function of $Y = \sum_{i=1}^n X_i$.

   Solution:

   The mgf of each X_i is given by $M_X(t) = \mathrm{E}(e^{tX}) = \int_0^1 e^{tx}\, dx = \frac{1}{t}(e^t - 1)$.

   Thus the mgf for Y is given by $M_Y(t) = \prod_{i=1}^n M_{X_i}(t) = \left[\frac{1}{t}(e^t - 1)\right]^n$.

3. Show that for X and Y independent random variables, that the mgf for $Z = X - Y$ is $M_Z(t) = M_X(t) \cdot M_Y(-t)$.

   Solution: $Z = X + (-Y)$ so the mgf for Z is $M_X(t) \cdot M_{-Y}(t)$ So we just need to know the mgf for $-Y$.

   $M_{-Y}(t) = \mathrm{E}(e^{t(-Y)} = \mathrm{E}(e^{(-t)Y} = M_Y(-t)$

   Recall that we have previously seen that $M_{aX}(t) = M_X(at)$ from Theorem 4.10.

4. If $X_k \sim \mathrm{Pois}(\lambda = \frac{1}{2^k})$ and are independent, prove that $Y = \sum_{k=1}^{\infty} X_k \sim \mathrm{Pois}(\lambda = 1)$.

   Solution:

   $M_{X_k}(t) = e^{\frac{1}{2^k}(e^t-1)}$. Thus $M_Y(t) = e^{\frac{1}{2}(e^t-1)} \cdot e^{\frac{1}{4}(e^t-1)} \cdot e^{\frac{1}{8}(e^t-1)} \cdots = e^{(e^t-1)\sum_{k=1}^{\infty}\frac{1}{2^k}} = e^{(e^t-1)}$.
   This is exactly the Poisson mgf with parameter $\lambda = 1$.