MATH 422 – SPRING 2020 – EXAM 1 || 100 POINTS || 50 MINUTES

Instructions: Show all work. No collaboration. Textbook and course notes allowed as references. Computational devices allowed.	Print Name ______________________

1. Let $X_i \sim \mathrm{N}(\mu, \sigma_1^2 = 4)$ for $i = 1, \ldots, 5$ and $Y_i \sim \mathrm{N}(\mu, \sigma_2^2 = 9)$ for $i = 1, \ldots, 8$. Calculate $\mathrm{P}(\overline{X} > \overline{Y})$.

2. Let $X_1, \ldots, X_n$ be uniformly distributed on $[\alpha, \beta]$. You are given that the minimum $X_{(1)}$ and maximum $X_{(n)}$ order statistics are biased estimators of α and β respectively and satisfy $\mathrm{E}(X_{(1)}) = \alpha + \frac{1}{n+1}(\beta - \alpha)$ and $\mathrm{E}(X_{(n)}) = \beta - \frac{1}{n+1}(\beta - \alpha)$. Show that $\hat{\mu} = \frac{1}{2}(X_{(1)} + X_{(n)})$ is an unbiased estimator of the mean $\mu = \frac{1}{2}(\alpha + \beta)$.

3. For $X_i \sim \mathrm{N}(\mu, \sigma^2)$, show that $\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is a consistent estimator of σ^2.

4. Let $X_i \sim \mathrm{N}(\mu, \sigma^2)$ be an i.i.d. sample. Consider the following estimator for a population mean.

$$\hat{\mu} = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i & \text{if } n \leq 100 \\ X_1 & \text{if } n > 100 \end{cases}$$

   Show that $\hat{\mu}$ is not consistent. *(Hint: Consistency requires $\mathrm{P}(|\hat{\theta} - \theta| > \epsilon) \to 0$ as $n \to 0$ for any "error tolerance" $\epsilon > 0$. Let $\epsilon = k\sigma$ be a fixed multiple of the standard deviation and calculate this probability as the sample size changes.)*

5. Consider the Pareto distribution $f(x) = \frac{\theta}{x^{n+1}}$ for $x > 1$ and parameter $\theta > 1$ and i.i.d. sample $X_1, \ldots, X_n$.
   (a) Find the *method of moments* and *maximum likelihood* estimators for θ.
   (b) Show that $Y_i = \ln X_i$ is exponentially distributed with *rate* θ. *(Hint: Show that $P(Y_i \leq y) = P(X_i \leq e^y) = F(y)$ where F is the exponential cdf.)*
   (c) You were given the formula for the MVUE for the exponential rate. This also gives you an estimator for the Pareto parameter. Argue that it is also the MVUE for the Pareto parameter. *(Note that you do not need to derive the MVUE for the exponential distribution rate as it was given to you in class. Deriving the MVUE for the exponential rate involves the gamma distribution and is a bit tricky. You just need to present some reasoning that it is indeed the MVUE for the pareto parameter. There is not a complicated calculation necessary.)*
   (d) Now use the above answers to derive the bias for the maximum likelihood estimator. Show that the MLE is asymptotically unbiased.

6. A recent poll of 900 likely voters for the general presidential election indicated 52% support for Biden. Estimate the true proportion of voters who vote for Biden with a 95% confidence interval.

7. Consider two groups of individuals who were given IQ tests. Group 1 had a mean score of 102 with standard deviation 13 and 7 people. Group 2 had a mean score of 117 with a standard deviation of 14 and 11 people. Assume that each group comes from a population which has normally distributed IQ scores. Construct a 95% confidence interval for the difference in the population mean IQ scores.