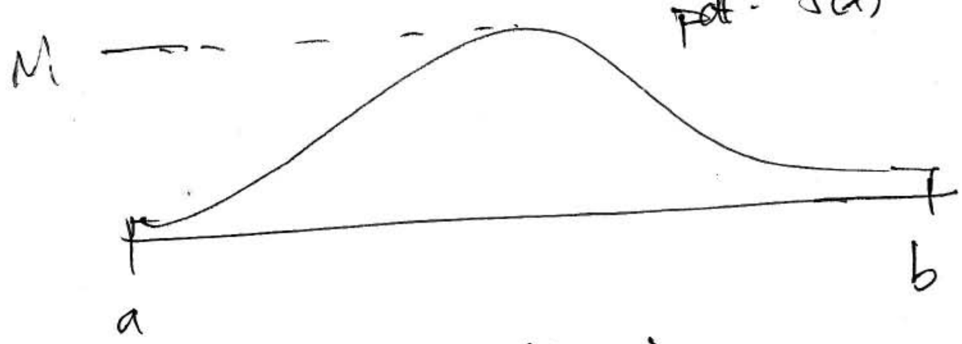
Sampling: ①

Rejection sampling:



w/ finite support

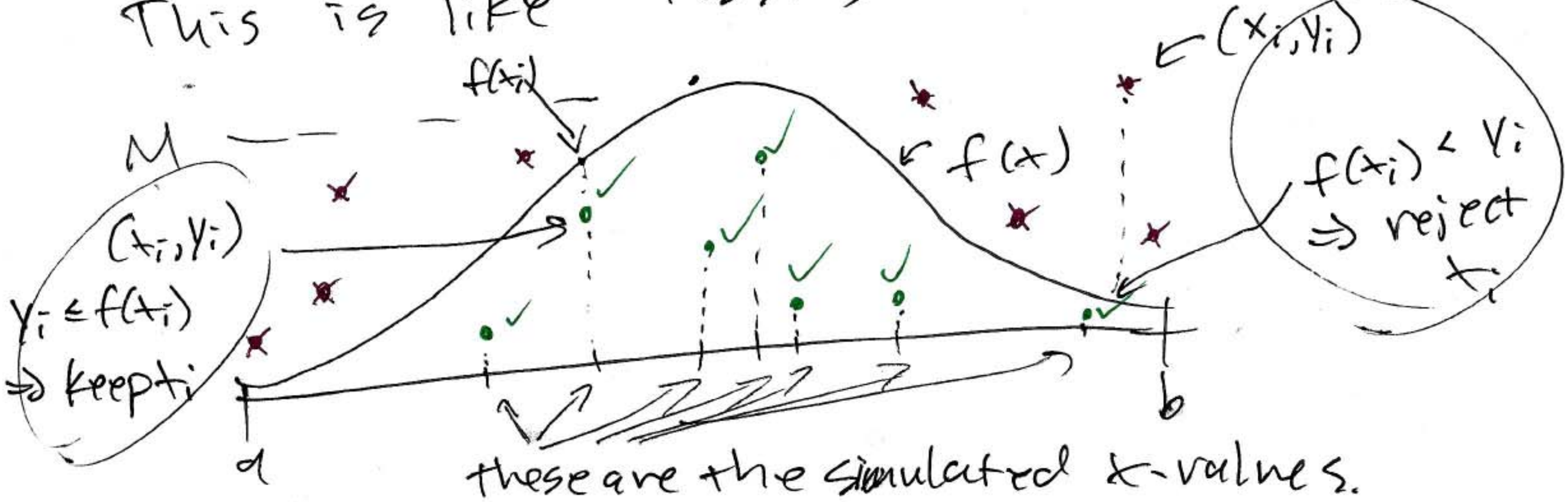
$f(x) = 0$ for $x < a$, $x > b$

$$M = \max_{[a,b]}\{f(x)\}$$

If we want a sample of size n,
we can generate n unif. x & y coords. indep.

$$\left.\begin{aligned} X &= \text{runif}(n, a, b) \\ Y &= \text{runif}(n, 0, M) \end{aligned}\right\} \text{plot as } (X, Y)$$

This is like "tossing darts" uniformly



these are the simulated x-values.

$\Rightarrow$ Keep all x_i with $f(x_i) \geq y_i$

$\Rightarrow$ don't get a sample of size n, but can, keep sampling until get desired sample size.

Rejection sampling.

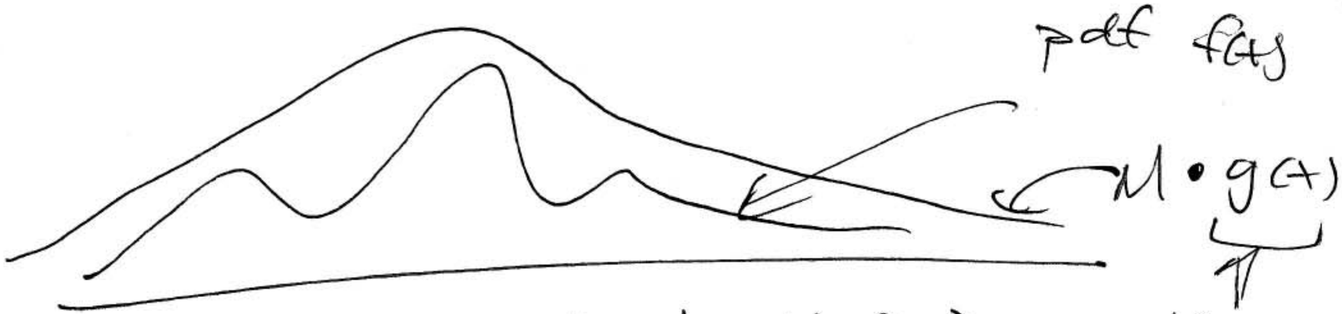
Approx. w/ pdf support is infinite

pick a & b s.t. $P(X<a)$, $P(X>b)$
very small

e.g. $f(x)=e^{-x}$, let $a=0, b=100$ or so.

Then sample as before.
This will be biased since $X>b$ not allowed,
but will still be a good fit.

Better more general rejection sampling



s.t. $M \cdot g(x) \geq f(x)$ $\forall x$

pdf we can sample from

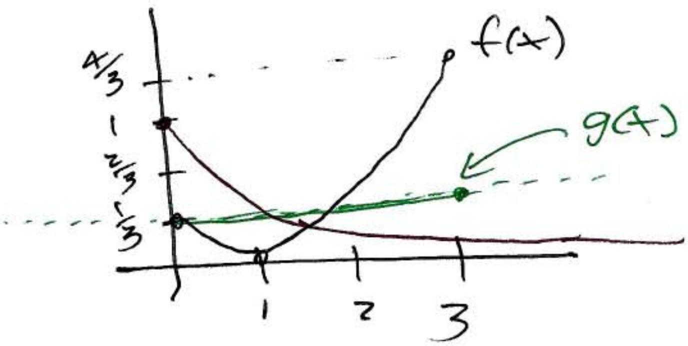
Algorithm.

1) simulate x from $g(x)$
2) simulate $u \sim U[0,1]$
3) If $u \leq \frac{f(x)}{M \cdot g(x)}$ keep x

   $u > \frac{f(x)}{M \cdot g(x)}$ reject x
4) Continue until desired sample size reached.

ex. $f(x) = \frac{1}{3}(x-1)^2 \quad x \in [0,3]$

*check L.O.T.P. $\int_0^3 f(x)dx = 1$

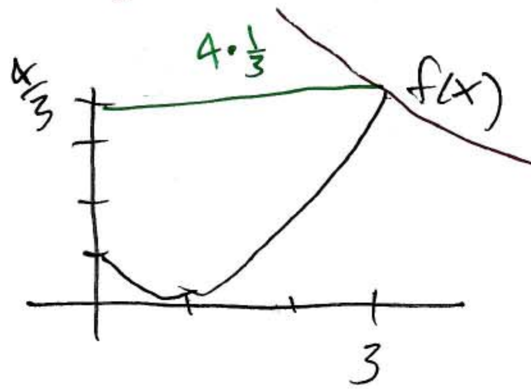


$\leftarrow g(x) = \frac{1}{3}$ on $[0,3]$ for $U[0,3]$

$\leftarrow g(x) = e^{-x}$ on $[0,\infty)$ for $Exp(\alpha = 1)$ (α ↑ rate)

1) $g(x) = \frac{1}{3}$ scale up by $M = 4$

2) $g(x) = e^{-x}$ scale up by $M = \frac{4}{3} \cdot e^3$



Now we have 2 options w/ $M \cdot g(x) \geq f(x) \quad \forall x \in [0,3]$

Algorithm: 1) $x = \text{runif}(n,0,3) \leftarrow \boxed{\text{sample from } g(x) = \frac{1}{3} \; U[0,3]}$

$u = \text{runif}(n,0,1)$

keep only x w/

$$u < \frac{f(x)}{M \cdot g(x)} = \frac{\frac{1}{3}(x-1)^2}{4 \cdot \frac{1}{3}}$$

$$\boxed{x\text{keep} = x\left[u < \tfrac{1}{3}(x-1)^2 / (4/3)\right]}$$

2) $x = \text{rexp}(n,1)$

$u = \text{runif}(n,0,1)$

keep only x w/ $x \leq 3$

& $u < \frac{f(x)}{M \cdot g(x)} = \frac{\frac{1}{3}(x-1)^2}{\frac{4}{3}e^3 \cdot e^{-x}}$

$x[u < f(x)/(M\,g(x))]$