MATH 413 – Real Analysis I – Fall 2020 – HW 01

Please answer the questions below and either turn in a paper copy in-person or make a quality scan into a single pdf and submit via email or Blackboard.

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1. Prove Part (f) of Theorem 1.7.2.

   Solution:

   Given that F is a field and $a \in F$, prove that $-(-a) = a$.

   We have that $a \in F$ implies $-(a) \in F$ which in turn implies $-(-a) \in F$ since F contains additive inverses for all its elements. Now we have that $a + (-a) = 0$ and we also have that $-(-a) + (-a) = 0$ since we are working with additive inverses. So we have that $-(-a) + (-a) = a + (-a)$, and that we can add a to both sides to get $(-(-a) + (-a)) + a = (a + (-a)) + a$. And by associativity we get $-(-a) + ((-a) + a) = a + ((-a) + a)$. Finally this simplifies to $-(-a) + 0 = a + 0$, and thus $-(-a) = a$ as desired.

2. Prove Part (b) of Theorem 1.7.4.

   Solution:

   Given $a, b, c \in F$ and $c < 0$, prove that $ac > bc$.

   By Theorem 1.7.4(a) we know that $-c > -0$ and from previous work that $-0 = 0$. Now we apply Axiom 12 to $-c$. This gives us that $a \cdot (-c) < b \cdot (-c)$. By commutivity, associativity, and Theorem 1.7.2(d), we can write this as $-(ac) < -(bc)$. But again by part (a) of this same theorem, this inequality is true if and only if $ac > bc$.

3. Section 1.7 Exercise 3 (b).

   Solution:

   Prove that $0 < 1$.

   By trichotomy (Axiom 9), we have $0 < 1$, $0 = 1$ or $0 > 1$. We cannot have $0 = 1$ since Axiom 6 says that the additive and multiplicative identities must be distinct. If $0 > 1$, then by Axiom 12 we have that for any $c > 0$, $0 \cdot c > 1 \cdot c = c$, but this implies that $0 > c$ which is a contradiction since c cannot be simultaneously greater than and less than zero. So the only possibility is that $0 < 1$.
   Now it could be that there are no such c satisfying $c \neq 1$ with $c > 0$. Since we have $1 \in F$, we automatically have $-1 \in F$, and if $1 < 0$, then $-1 > 0$, so that we do indeed have at least one "positive" element in F. Our proof implies that this positive element is also negative, which is a contradiction.

4. Section 1.7 Exercise 5.

   Solution:

   Prove that if $r \geq 1$ then $r^2 > r$ and $\frac{1}{r^2} \leq \frac{1}{r}$.

   We have that $r \geq 1 > 0$, thus by Axiom 12 and $r > 0$ we have that $r \cdot r \geq 1 \cdot r$.
   Now, since $r \geq 1 > 0$ we also have that $\frac{1}{r} > 0$. So again we can apply Axiom 12 and multiply the inequality $r \geq 1$ across by $\frac{1}{r}$ twice. This gives $r \cdot \frac{1}{r} \geq 1 \cdot \frac{1}{r}$ and $r \cdot \frac{1}{r} \cdot \frac{1}{r} \geq 1 \cdot \frac{1}{r} \cdot \frac{1}{r}$. Simplifying gives the desired result.