MATH 413 – Real Analysis I – Fall 2020 – HW 02

Please answer the questions below and either turn in a paper copy in-person or make a quality scan into a single pdf and submit via email or Blackboard.

1. Finish the proof of Theorem 1.8.4(b), and prove Parts (c) and (d) as well.

   Solution:

   1.8.4(b). We have that $a < b \iff a^2 < b^2$. Now let us show that $a < b \iff \sqrt{a} < \sqrt{b}$, and this will complete the proof.

   Let $k = \sqrt{a}$ and $s = \sqrt{b}$. We know that $k, s \in \mathbb{R}$ and that $k, s > 0$.

   Note that we haven't proved this in class, nor is it proved in my notes, but I did give you some supplementary notes on roots and exponents that say that for any positive real number $a > 0$, $\sqrt{a}$ is also a real number. By definition of "$\sqrt{\ }$" we have that $\sqrt{a} > 0$ when $a > 0$. If we wanted the negative square root, we would have to write "$-\sqrt{a}$".

   Now we apply what has already been proved. $k, s \in \mathbb{R}$ and $k, s > 0$ implies that

   $$k < s \iff k^2 < s^2$$

   but now we can put back in the definitions of k, s in terms of a, b to get

   $$a < b \iff \sqrt{a} < \sqrt{b}$$

   and we are done.

   To be more clear, $0 < \sqrt{a} < \sqrt{b}$ implies that $0 < (\sqrt{a})^2 < (\sqrt{b})^2$ and $0 < (\sqrt{a})^2 < (\sqrt{b})^2$ implies that $0 < \sqrt{a} < \sqrt{b}$.

   1.8.4(c). Note that $0 \leq (\sqrt{a} - \sqrt{b})^2 = a + b - 2\sqrt{ab}$. Now add $\sqrt{ab}$ and divide by 2.

   1.8.4(d). Note that $a + b = \sqrt{(a+b)^2} = \sqrt{a^2 + b^2 + 2ab} \geq \sqrt{a^2 + b^2}$ since $ab \geq 0$. We are actually applying Theorem 1.8.4(b) here to $a^2 + b^2 \leq a^2 + b^2 + 2ab$ which implies $\sqrt{a^2 + b^2} \leq \sqrt{a^2 + b^2 + 2ab} = a + b$.

2. Prove Theorem 1.8.5(c).

   Solution:

   We want to show that $|a| \geq b$ if and only if $a \leq -b$ or $a \geq b$.

   ($\Rightarrow$) Assume that $|a| \geq b$. Then either $a \geq 0$ or $a < 0$. If $a \geq 0$, then $a = |a| \geq b$. Hence $a \geq b$. If $a < 0$, then $-a = |a| \geq b$. Hence $-a \geq b$, which is identical to $a \leq -b$.

   ($\Leftarrow$) Assume that either $a \geq b$ or $a \leq -b$ is true. Also assume that $b > 0$. Note that $b > 0$ means that only one is true, and that $b \leq 0$ means that both can be true simultaneously. Also, if $b \leq 0$, then $|a| \geq b$ is trivial.

   Again we have that $a \geq 0$ or $a < 0$. If $a \geq 0$ then $|a| = a$ and $a \geq b > 0$ gives $|a| \geq b > 0$. Since we are assuming that $b > 0$ and $a \geq 0$, then we will never have that $a \leq -b$.

   Now if $a < 0$ (again keeping with the assumption that $b > 0$, hence $a \geq b$ is impossible), then $a \leq -b$ gives $-a \geq b$, and thus $|a| = -a \geq b$.

3. Prove Corollary 1.8.6(b).

   Solution:

   By Corollary 1.8.6(a) we have $|a-b| \geq |a|-|b|$ and also $|a-b| = |b-a| \geq |b|-|a| = -(|a|-|b|)$. Thus we have that $|a-b| \geq |a|-|b|$ and $|a-b| \geq -(|a|-|b|)$. Now notice that $\Big||a|-|b|\Big| = \pm(|a|-|b|)$. So that in either case we have $|a-b| \geq \Big||a|-|b|\Big|$.

4. Section 1.8, Exercise 14 (c) and (e).

   Solution:

   14(c). This is something that I have essentially given to you in my supplementary notes on roots and exponents. As far as i can tell, Kosmala never formally introduces square roots. On p. 11 is the first time "$\sqrt{}$" appears, and on p. 13 it is stated that $|x| = \sqrt{x^2}$. So you can essentially reference that and be done. I'll do a bit of reasoning below though.

   You can argue that $(\sqrt{a^2})^2 = a^2$ by the definition of "$\sqrt{}$". So the only options are $\sqrt{a^2} = a$ or $\sqrt{a^2} = -a$.

   If $a \geq 0$, then $|a| = a$, and if $a < 0$, then $|a| = -a$. So if $\sqrt{a} = -a$, then $\sqrt{a} = 0$ or $\sqrt{a} < 0$, and we wish to specify that $\sqrt{a} \geq 0$ for any $a \geq 0$. Thus $\sqrt{a^2} = a = |a|$.

   If $a < 0$, then $-a > 0$ and $\sqrt{a^2} = \sqrt{(-a)^2} = -a = |a|$.

5. Section 1.8, Exercise 18.

   Solution:

   (a) For all $a, b, c \in \mathbb{R}$ we have that $|a-b| = |a-c+c-b| \leq |a-c| + |c-b| = |a-c| + |b-c|$.

   (b) Let $a < b < c$, then $b-a > 0$, $c-a > 0$, $c-b > 0$. Note that $|x-y| = |y-x|$ for all $x, y \in \mathbb{R}$.
   Now we reason that $|a-c| = |c-a| = c-a = c-b+b-a = |c-b| + |b-a| = |a-b| + |b-c|$. Note that removing and adding the absolute values was ok since we were working with strictly positive quantities.

6. Section 2.1, Exercise 2 (d) and (k).

Solution:

2.1#2(d). Multiply $a_n = \frac{n}{2n+\sqrt{n}}$ on top and bottom by $\frac{1}{n}$ to get $\dfrac{1}{2+\frac{1}{\sqrt{n}}}$. Now you should intuit that $\frac{1}{\sqrt{n}}$ converges to zero, and thus you should start to believe that $a_n \to \frac{1}{2}$. Now we will present a formal argument.

Given any $\epsilon > 0$, choose any $n^* > \frac{1}{4\epsilon^2}$. Then for any $n \geq n^*$ we have that

$$\begin{aligned}\left|\frac{n}{2n+\sqrt{n}} - \frac{1}{2}\right| &= \left|\frac{-\sqrt{n}}{2(2n+\sqrt{n})}\right| \\ &= \frac{\sqrt{n}}{2(2n+\sqrt{n})} \\ &= \frac{1}{2(2\sqrt{n}+1)} \\ &= \frac{1}{4\sqrt{n}} \\ &= \frac{1}{4\sqrt{n}^*} \\ &= \frac{1}{4\sqrt{\frac{1}{4\epsilon^2}}} \\ &= \epsilon\end{aligned}$$

Solution:

2.1#2(k).

$$a_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \frac{1}{n} & \text{if } n \text{ is even} \end{cases}$$

This sequence diverges. Since it hits 1 infinitely often, if it has a limit, the limit would need to be 1. But it will also be very close to 0 infinitely often. So its limit should be 0 or at least some number very close to zero. This can all not be true if it is indeed a convergent sequence.

Clearly if $a_n \to A$, then $A \geq 0$ since $a_n > 0$ for all n. It cannot possibly converge to a negative number. Then we need to show it cannot converge to any non-negative number.

Here is our strategy: (1) show a_n cannot converge to any $A < 0$, (2) show a_n cannot converge to zero, and then finally (3) show a_n cannot converge to any $A > 0$.

(1) Assume $A < 0$ so that $B = -A > 0$ and pick any $\epsilon < B$. We know infinitely many such choices for ϵ exist since $0 < B$ ($\epsilon = \frac{B}{2}$ works, for example). Now $|a_n - A| = a_n + B$ since $a_n > 0$ and $-A = B > 0$. We have that $a_n + B = 1 + B > B > \epsilon$ when n is odd, and $a_n + B = \frac{1}{n} + B > B > \epsilon$ when n is even. Then no matter what we set as our cut-off n^*, there will be even and odd $n \geq n^*$ such that $|a_n - A| > B > \epsilon$. So a_n cannot converge to $A < 0$. We write this as $a_n \not\to A < 0$.

(2) Assume $A = 0$. Then $|a_n - A| = a_n$ since $a_n \geq 0$ for all n. Pick any $\epsilon > 0$ but less than one, e.g. $\epsilon = \frac{1}{2}$. Then no matter what we pick as our cut-off n^* there will be odd $n \geq n^*$ such that $|a_n - A| = a_n = 1 > \epsilon = \frac{1}{2}$. Thus $a_n \not\to A = 0$.

(3) Now assume $A > 0$. Pick any $\epsilon > 0$ such that $0 < \epsilon < A$ such as $\epsilon = \frac{A}{2}$. We have that $0 < A - \epsilon$. Then there exists n^* such that $0 < \frac{1}{n^*} < A - \epsilon < A$. *(This is given by Theorem 1.7.9(c), but you no longer need to reference theorems, definitions, or axioms from Chapter 1. However, make sure you really are sure about what you are doing!)* Then for any $n \geq n^*$ with n even we have that

$$\begin{aligned}
|a_n - A| &= \left|\frac{1}{n} - A\right| && \text{(since } n \text{ is even)} \\
&= A - \frac{1}{n} && \left(\text{since } n \geq n^* \text{ and } \frac{1}{n} \leq \frac{1}{n^*} < A\right) \\
&\geq A - \frac{1}{n^*} && \left(\text{since } n \geq n^* \text{ and } \frac{1}{n} \leq \frac{1}{n^*} \Rightarrow -\frac{1}{n} \geq -\frac{1}{n^*}\right) \\
&> A - (A - \epsilon) && \left(\text{since } \frac{1}{n^*} < A - \epsilon \Rightarrow -\frac{1}{n^*} > -(A - \epsilon)\right) \\
&= \epsilon
\end{aligned}$$

Thus as long as $0 < \epsilon < A$, no matter how large we choose our cut-off n^*, there will be even $n \geq n^*$ such that $|a_n - A| > \epsilon$. Therefore $a_n \not\to A > 0$.

We conclude that a_n cannot converge to any real number.