MATH 413 – Real Analysis I – Fall 2020 – HW 04

Please answer the questions below and either turn in a paper copy in-person or make a quality scan into a single pdf and submit via email or Blackboard.

Name the pdf file: `hw04_math413_lastname.pdf` with "lastname" of course replaced by your last name.

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1. Section 2.3, Exercise 3(a)
2. Section 2.4, Exercise 2. Also: give an example of a converging sequence that does attain its maximum. You do not need to prove your results, but give some argument.
3. Section 2.4, Exercise 5
4. Consider the sequence defined by $a_1 = 1$ and $a_{n+1} = \sqrt{a_n + 1} + 1$ for all $n \geq 1$. Prove that this sequence converges and find its limit. *(Hint: use monotone convergence. Induction may be helpful to show the sequence is bounded and increasing.)*
5. Section 2.5, Exercise 11
6. (a) Consider the sequence

$$a_n = \begin{cases} (-1)^n + 3 & \text{when } n \text{ is a multiple of } 5 \\ \frac{1}{n} & \text{otherwise} \end{cases}$$

   Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

   (b) Now consider the subsequence $b_n = a_{5n}$ for $n \in \mathbb{N}$. Find $\limsup_{n\to\infty} b_n$ and $\liminf_{n\to\infty} b_n$.

7. (OPTIONAL) Prove the following theorem about "swapping the order" of strictly increasing convergent sequences.

   Theorem. Let a_n and b_n be two strictly increasing sequences converging to the same limit and satisfying $a_n < b_n$ for all n. Prove that the exists a subsequence of a_n, $c_n = a_{f(n)}$ that satisfies $b_n < c_n$ for all n.