MATH 413 – Real Analysis I – Fall 2020 – HW 03

Please answer the questions below and either turn in a paper copy in-person or make a quality scan into a single pdf and submit via email or Blackboard.

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1. Prove that $\frac{n^2}{n^2-101}$ converges.

   Solution:

   Given any $\epsilon > 0$, choose any natural number $n^* > \max\{10, \sqrt{\frac{101}{\epsilon}+101}\}$. Then for any $n \geq n^*$ we have that

$$\begin{aligned} \left|\frac{n^2}{n^2-101} - 1\right| &= \left|\frac{101}{n^2-101}\right| \\ &= \frac{101}{n^2-101} && \text{(since } n \geq n^* > 10) \\ &\leq \frac{101}{(n^*)^2-101} && \text{(since } n \geq n^* > 10) \\ &< \frac{101}{\left(\sqrt{\frac{101}{\epsilon}+101}\right)^2 - 101} && \text{(since } n^* > \sqrt{\frac{101}{\epsilon}+101}) \\ &= \epsilon \end{aligned}$$

   Thus we can make $\left|\frac{n^2}{n^2-101} - 1\right| < \epsilon$ eventually true regardless of the choice for $\epsilon > 0$. Therefore $\frac{n^2}{n^2-101}$ converges to 1.

2. Section 2.1, Exercise 8.

   Solution:

   Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences with $b_n \to 0$. If there exist constants A and k and a positive integer n^* such that $|a_n - A| < k|b_n|$ for all $n \geq n^*$, prove that the sequence $\{a_n\}$ must converge to A.

   Since $b_n \to 0$ and that there exist constants A and k and a positive integer n^* such that $|a_n - A| < k|b_n|$ for all $n \geq n^*$. This says that the sequence is sort of getting close to A since b_n is converging to zero, then $|b_n|$ is eventually getting small. Thus $k|b_n|$ is eventually getting small.

   Proof. Let $\epsilon > 0$. Assume Since $b_n \to 0$. Now suppose that there exist constants A and k and a positive integer n_1 such that $|a_n - A| < k|b_n|$ for all $n \geq n_1$. Since $b_n \to 0$ and $k > 0$, then for any $\epsilon > 0$ we can find an n_2 such that $n \geq n_2$ implies that $|b_n| < \frac{\epsilon}{k}$ (because ϵk is a positive number). Now choose any $n^* \geq \max\{n_1, n_2\}$.

   Now for all $n \geq n^*$ we have that $|a_n - A| < k|b_n| < k \cdot \frac{\epsilon}{k} = \epsilon$. Thus $a_n \to A$.

3. Section 2.1, Exercise 11. No need to prove your example, just give an example. Provide at least some explanation of why it is bounded but doesn't converge.

<u>*Solution:*</u>

Example 1: $a_n = (-1)^n a + b$ for any $a, b \in \mathbb{R}$ such that $a \neq 0$ is bounded but not convergent since $|a_n| \leq |a| + |b|$ and a_n will hit $a+b$ and $-a+b$ infinitely many times and these quantities are distinct since $a \neq 0$.

Example 2: $a_n = \sin(n)$. This is a complicated sequence actually! Intuitively, you might guess that it bounces around between -1 and 1 and so it is bounded. Proving it doesn't converge is actually very complicated though. In fact, $\{\sin(n) \mid n \in \mathbb{N}\}$ is **dense** in $[-1, 1]$. This means that we can find an n where $\sin(n)$ is as close as we want to any real number in $[-1, 1]$. This implies that a_n will never converge as it will get arbitrarily close to every number in $[-1, 1]$ infinitely many times! Do an online search for "density of sine function".

Example 3: Here is a funny example just to show that we can construct sequences in interesting ways:

$$a_n = \begin{cases} 1 & \text{when } n \text{ is an odd prime} \\ \frac{1}{n} & \text{for } n \text{ even} \\ n^2 & \text{for } n \text{ odd, but not prime, and } n \leq 100 \\ 0 & \text{for } n \text{ odd, but not prime, and } n > 100, \end{cases}$$

We certainly have $0 \leq a_n \leq 100^2$ so a_n is bounded, but there are infinitely primes so $a_n = 1$ for infinitely many n, and there are infinitely many non-prime odd numbers beyond 100 as well, thus $a_n = 0$ infinitely many times. Therefore a_n cannot converge.

4. Section 2.1, Exercise 12.

<u>*Solution:*</u>

If the sequence a_n converges to a nonzero constant A and $a_n \neq 0$, for any n, prove that the sequence $\frac{1}{a_n}$ is bounded.

Proof. Assume $a_n \to A \neq 0$ and that $a_n \neq 0$ for all $n \in \mathbb{N}$. Then we know that $\frac{1}{a_n}$ is defined (and nonzero) for all n.

Case I: Assume $A > 0$ and let $\epsilon = \frac{A}{2} > 0$. Then we know there is an n^* such that $n \geq n^*$ implies that $|a_n - A| < \frac{A}{2}$. This implies that $0 < \frac{A}{2} < a_n < 3\frac{A}{2}$ for all $n \geq n^*$. In other words we have that $0 < \frac{A}{2} < |a_n| < 3\frac{A}{2}$ for all $n \geq n^*$ and dividing and rearranging gives $\frac{1}{3\frac{A}{2}} < \frac{1}{|a_n|}$ and $\frac{1}{|a_n|} < \frac{1}{\frac{A}{2}}$, i.e. that $\frac{1}{3\frac{A}{2}} < \frac{1}{|a_n|} < \frac{1}{\frac{A}{2}} = \frac{2}{A}$. Thus we have that $\left|\frac{1}{a_n}\right| < \frac{2}{A}$ for all $n \geq n^*$.

Now we know that $\{|a_1|, |a_2|, \ldots, |a_{n^*-1}|\}$ is a finite list of positive numbers and thus is bounded from above and below. Let $m = \min_{n<n^*}\{|a_n|\}$ and $M = \max_{n<n^*}\{|a_n|\}$. Note that $m > 0$ and $M > 0$ since $|a_n| \neq 0$ for all n. So we have that $0 < m \leq |a_n| \leq M$ for $n = 1, 2, \ldots, n^* - 1$. This also gives that $0 < \frac{1}{M} \leq \left|\frac{1}{a_n}\right| \leq \frac{1}{m}$ for $n = 1, 2, \ldots, n^* - 1$.

So we have that $\left|\frac{1}{a_n}\right| \leq \frac{2}{A}$ for $n \geq n^*$ and that $\left|\frac{1}{a_n}\right| \leq \frac{1}{m}$ for $n < n^*$. Let $K = \max\{\frac{2}{A}, \frac{1}{m}\}$. Note that $K > 0$, and we have that $\left|\frac{1}{a_n}\right| \leq K$ for all $n \in \mathbb{N}$. Thus $\frac{1}{a_n}$ is bounded.

5. Prove Theorem 2.2.1, part (d).

Solution:

First note that for any $p \in \mathbb{N}$ we have that

$$x^p - y^p = (x - y) \sum_{k=0}^{p-1} x^{p-1-k} y^k$$

which can be proved by induction.

Assume that $a_n \to A$. Then we know that a_n is bounded by some M, i.e. that $|a_n| \leq M$ for all n. Then we have that

$$\left| \sum_{k=0}^{p-1} a_n^{p-1-k} A^k \right| \leq \sum_{k=0}^{p-1} |a_n|^{p-1} |A|^{p-1} \leq \sum_{k=0}^{p-1} M^{p-1} |A|^{p-1}$$

Let $C = \sum_{k=0}^{p-1} M^{p-1} |A|^{p-1}$.

Now let $\epsilon > 0$, thus we have that $\frac{\epsilon}{C} > 0$. Since $a_n \to A$ we can find an $n^* \in \mathbb{N}$ such that for any $n \geq n^*$ we have that $|a_n - A| < \frac{\epsilon}{C}$.
Now finally for any $n \geq n^*$ we have that

$$\begin{aligned} |(a_n)^p - A^p| &= |a_n - A| \cdot \left| \sum_{k=0}^{p-1} a_n^{p-1-k} A^k \right| \\ &\leq |a_n - A| \cdot C \\ &< \frac{\epsilon}{C} \cdot C = \epsilon \end{aligned}$$

This proves that a_n^p converges to A^p.

Note that we can strengthen this result to say that $a_n \to A$ if and only if $a_n^r \to A^r$ for any $r \in \mathbb{R}$ with $r \geq 0$ when $A \geq 0$ and $a_n \geq 0$ for all n. If $r < 0$, then we also require that $a_n \neq 0$ for all n. This is a bit trickier of an argument, but we will see later that it is related to the fact that power functions like $f(x) = x^r$ are continuous.

6. Section 2.2, Exercise 19.

   Solution:

   Consider the sequences a_n and b_n, where $b_n = \frac{a_n+1}{a_n-1}$. If b_n converges to zero, prove that a_n converges to -1.

   Proof. Note that solving for a_n gives $a_n = \frac{b_n+1}{b_n-1}$. Since $b_n \to 0$, we can choose an n_1 such that $|b_n| < \frac{1}{2}$ for all $n \geq n_1$. This implies that $-\frac{1}{2} < b_n < \frac{1}{2}$ and thus $-\frac{3}{2} < b_n - 1 < -\frac{1}{2}$, or in other words that $\frac{1}{2} < |b_n - 1| < \frac{3}{2}$. Also this implies that $\frac{2}{3} < \frac{1}{|b_n-1|} < 2$.

   Now let $\epsilon > 0$ and choose some n_2 such that for all $n \geq n_2$ we have $|b_n| < \frac{\epsilon}{4}$. Now choose any $n^* \geq \max\{n_1, n_2\}$. Then we have that for any $n \geq n^*$

$$\begin{aligned}
|a_n - (-1)| &= |a_n + 1| \\
&= \left|\frac{b_n+1}{b_n-1} + 1\right| \\
&= \left|\frac{2b_n}{b_n-1}\right| \\
&= 2 \cdot |b_n| \cdot \left|\frac{1}{b_n-1}\right| \\
&< 2 \cdot \frac{\epsilon}{4} \cdot 2 = \epsilon
\end{aligned}$$

   Thus no matter what we are given for $\epsilon > 0$, we can choose a cut-off n^* such that $|a_n - (-1)| < \epsilon$ when $n \geq n^*$. This implies that $a_n \to -1$.

7. You are given that x_n converges to x and that y_n is bounded. Prove that $x_n + \frac{y_n}{n}$ converges to x.

Solution:

Since y_n is bounded, then there is a real number M such that $|y_n| \leq M$ for all n. Thus by the triangle inequality we have that

$$\left|x_n + \frac{y_n}{n} - x\right| = \left|(x_n - x) + \frac{y_n}{n}\right| \leq |x_n - x| + \left|\frac{y_n}{n}\right| < \epsilon + \frac{M}{n}$$

Now we just need to figure out how to get rid of the $\frac{M}{n}$. we can do this as follows.

Proof. Let $\epsilon > 0$. We have that y_n is bounded by $M > 0$. Choose any n_1 such that $\frac{M}{n_1} < \epsilon$. Note that now $n \geq n_1$ implies that $\frac{M}{n} \leq \frac{M}{n_1} < \epsilon$.

Now we also have that $0 < \epsilon - \frac{M}{n_1}$. So choose some n_2 such that $n \geq n_2$ implies that $|x_n - x| < \epsilon - \frac{M}{n_1}$. We can do this since $0 < \epsilon - \frac{M}{n_1}$ and $x_n \to x$.

Now choose any $n^* \geq \max\{n_1, n_2\}$. Thus for any $n \geq n^*$ we have that

$$\begin{aligned}
\left|\left(x_n + \frac{y_n}{n}\right) - x\right| &= \left|(x_n - x) + \frac{y_n}{n}\right| \\
&\leq |x_n - x| + \left|\frac{y_n}{n}\right| && \text{(triangle inequality)} \\
&\leq |x_n - x| + \frac{M}{n} && \text{(since } |y_n| \leq M\text{)} \\
&< \left(\epsilon - \frac{M}{n_1}\right) + \frac{M}{n} && \text{(since } n \geq n^* \geq n_2\text{)} \\
&\leq \left(\epsilon - \frac{M}{n_1}\right) + \frac{M}{n_1} && \text{(since } n \geq n^* \geq n_1\text{)} \\
&= \epsilon
\end{aligned}$$

Thus we have that $x_n + \frac{y_n}{n} \to x$.

8. Finish the proof of Theorem 2.3.6.

Solution:

THEOREM 2.3.6. Consider a sequence a_n, where $a_n > 0$ for all n. Then a_n diverges to $+\infty$ if and only if the sequence $\frac{1}{a_n}$ converges to zero.

The textbook contains a proof of the ($\Rightarrow$) direction, that

$$a_n \to +\infty \quad \Longrightarrow \quad \frac{1}{a_n} \to 0.$$

We prove the other direction here,

$$a_n \to +\infty \quad \Longleftarrow \quad \frac{1}{a_n} \to 0.$$

Proof. ($\Leftarrow$) Suppose $\frac{1}{a_n} \to 0$. Note that $a_n \neq 0$ is required for the sequence $\frac{1}{a_n}$ to be defined. Let $M > 0$, then $\epsilon = \frac{1}{M} > 0$ also. Then there is an n^* such that $n \geq n^*$ implies that $\frac{1}{a_n} = \left|\frac{1}{a_n}\right| < \epsilon = \frac{1}{M}$. This shows that $n \geq n^*$ implies that $|a_n| > M$.

Since we have just show that for any arbitrary positive real number M, we can show that, eventually, the sequence $a_n > M$. This means that a_n diverges to $+\infty$.