MATH 413 – QUIZ 1 – Fall 2020 || 50 POINTS

SOLUTIONS

1. (5 pts) Consider the number system $\widetilde{\mathbb{R}} = \mathbb{R} \setminus \{\sqrt{2}\}$ with mathematical operations the same as for $\mathbb{R}$. Is this number system a *complete ordered field*? Explain why or why not. You do not need to write a proof, but you should show some careful reasoning.

Solution:

Completeness requires that any nonempty subset of $\widetilde{\mathbb{R}}$ that is bounded above has a supremum that is also in $\widetilde{\mathbb{R}}$. We have already seen that $S = \{q \in \mathbb{Q} \mid 0 < q^2 < 2\} \subset \mathbb{Q} \subset \mathbb{R}$ is nonempty and bounded above and that $\sup S = \sqrt{2} \in \mathbb{R}$. Note that $S \subset \widetilde{\mathbb{R}}$ also! Removing $\sqrt{2}$ from $\mathbb{R}$ to create $\widetilde{\mathbb{R}}$ did not affect set S at all! So S is still nonempty and bounded above as a subset of $\widetilde{\mathbb{R}}$, thus if $\widetilde{\mathbb{R}}$ is complete, then $\sup S \in \widetilde{\mathbb{R}}$. But we know that if $b = \sup S$, then $b^2 = 2$ and $b > 0$. Our modified number system $\widetilde{\mathbb{R}}$ has a "hole" at $\sqrt{2}$ though so that there is no $b \in \widetilde{\mathbb{R}}$ with $b > 0$ such that $b^2 = 2$. Of course we still have $-\sqrt{2} \in \widetilde{\mathbb{R}}$, but that is of no consequence. Since we have a nonempty and bounded above subset of $\widetilde{\mathbb{R}}$ whose supremum is not a member of $\widetilde{\mathbb{R}}$, we conclude that $\widetilde{\mathbb{R}}$ is **NOT** complete.

Note that we did not even discuss whether or not $\widetilde{\mathbb{R}}$ even satisfies all field and order axioms. In fact $\widetilde{\mathbb{R}}$ is not even an ordered field at all! Note that $\sqrt{2} - 1, 1 \in \widetilde{\mathbb{R}}$ and thus $\sqrt{2} - 1 + 1$ should be in $\widetilde{\mathbb{R}}$ if we wanted it to satisfy the field axioms, but this is false! So our "number system" $\widetilde{\mathbb{R}}$ is not a field even!

This should give you some sense of why we need every single irrational number and cannot discard any at all in order to retain completeness. Even just appending a single irrational number on to $\mathbb{Q}$ will require us to include many others in order to keep the field axioms satisfied alone. E.g. consider $\mathbb{Q} \cup \{r\}$ for some $r \in \mathbb{R} \setminus \mathbb{Q}$. To make this a field, we need to include $q \pm r$ and $q \cdot r$ for all $q \in \mathbb{Q}$, and we also need to include the multiplicative and additive inverses of these as well, $\pm\frac{1}{q \pm r}$ and $q \cdot \frac{1}{r}$, etc. Note that when we multiply or add an irrational r and a rational q, the result is irrational, i.e. in addition to including r, we need to include a countable infinity of other irrationals. And this may then become a field, but it will not be complete. Completeness requires all irrationals to be included.

2. (15 pts) Prove that $a_n = \frac{2n+1}{n+\sqrt{n}}$ converges.

Solution:

This problem is actually a bit tricky. We first intuit that $a_n \to 2$. And that $|a_n - 2| = \left|\frac{1-2\sqrt{n}}{n+\sqrt{n}}\right|$. Now we must try to simplify this and get is less than ϵ. It might also be worth noting that even though we have a subtraction in the numerator, it is never 0 and is strictly negative so that we can remove t he absolute value bars if we swap the order: $\left|\frac{1-2\sqrt{n}}{n+\sqrt{n}}\right| = \frac{2\sqrt{n}-1}{n+\sqrt{n}}$.

Note that we can make the denominator smaller to get

$$\left|\frac{1-2\sqrt{n}}{n+\sqrt{n}}\right| < \left|\frac{1-2\sqrt{n}}{n}\right|$$

and that by the triangle inequality we have

$$\left|\frac{1-2\sqrt{n}}{n}\right| \leq \frac{1}{n} + \frac{2}{\sqrt{n}}$$

Finally we note that $\frac{1}{n} \leq \frac{1}{\sqrt{n}}$ for all n giving us that $\frac{1}{n} + \frac{2}{\sqrt{n}} \leq \frac{1}{\sqrt{n}} + \frac{2}{\sqrt{n}} = \frac{3}{\sqrt{n}}$.

Putting this all together gives us:

$$|a_n - 2| = \left|\frac{1-2\sqrt{n}}{n+\sqrt{n}}\right| < \left|\frac{1-2\sqrt{n}}{n}\right| \leq \frac{1}{n} + \frac{2}{\sqrt{n}} \leq \frac{1}{\sqrt{n}} + \frac{2}{\sqrt{n}} = \frac{3}{\sqrt{n}}$$

and $\frac{3}{\sqrt{n}} < \epsilon$ when $n > \frac{9}{\epsilon^2}$.

Now for our official proof:

Proof. Let $\epsilon > 0$ and choose any $n^* > \frac{9}{\epsilon^2}$. Then for any $n \geq n^*$ we have that

$$\begin{aligned} |a_n - 2| &= \left|\frac{2n+1}{n+\sqrt{n}} - 2 \cdot \frac{n+\sqrt{n}}{n+\sqrt{n}}\right| = \left|\frac{1-2\sqrt{n}}{n+\sqrt{n}}\right| \\ &< \left|\frac{1-2\sqrt{n}}{n}\right| \leq \frac{1}{n} + \frac{2}{\sqrt{n}} \\ &\leq \frac{1}{\sqrt{n}} + \frac{2}{\sqrt{n}} = \frac{3}{\sqrt{n}} \\ &\leq \frac{3}{\sqrt{n^*}} < \frac{3}{\sqrt{\frac{9}{\epsilon^2}}} = \epsilon \end{aligned}$$

Thus for any $\epsilon > 0$, we can choose a cut-off index n^* such that $n \geq n^*$ implies that $|a_n - 2| < \epsilon$. This shows that $a_n \to 2$. ∎

Another, simpler in some ways but somewhat requiring subtle reasoning, argument is that $\left|\frac{1-2\sqrt{n}}{n+\sqrt{n}}\right| = \frac{2\sqrt{n}-1}{n+\sqrt{n}} < \frac{2\sqrt{n}}{n+\sqrt{n}} = \frac{2}{\sqrt{n}+1}$ and then choose any $n^* > \left(\frac{2}{\epsilon} - 1\right)^2$. It would be worthwhile to check that this works for any $\epsilon > 0$ even though there is subtraction involved. Note that $\sqrt{\left(\frac{2}{\epsilon} - 1\right)^2} = 1 - \frac{2}{\epsilon}$ when $\epsilon \geq 2$, and that, in this case, $\frac{\epsilon}{\epsilon - 1} < \epsilon$ since $\epsilon - 1 \geq 1$. To sidestep this issue, we could go by the standard argument requiring that $\epsilon < 2$ and then to note that $\frac{2\sqrt{n}-1}{n+\sqrt{n}}$ is bounded above by 1. This gives that $|a_n - 2| = \frac{2\sqrt{n}-1}{n+\sqrt{n}} < \epsilon$ for all n when $\epsilon \geq 2$.

3. (15 pts) Prove that $a_n = \frac{2n^3+1}{n+1}$ diverges to infinity.

Solution:

Intuitively we can believe that this sequence behaves like $2n^2$ for large n. There are a variety of ways to get a_n bounded from below by a "nice enough" diverging sequence.
Consider that $b_n \leq \frac{2n^3+1}{n+1}$ if and only if $b_n(n+1) \leq 2n^3+1$. Let's try to find a nice b_n that works.

Try $b_n = 2n^2$. Then we want $2n^3 + 2n^2 \leq 2n^3 + 1$ giving $2n^2 \leq 1$. That doesn't work!

Try $b_n = n^2$. Then we want $n^3+n^2 \leq 2n^3+1$ giving $n^2 \leq n^3+1$. This is true for all n, so by comparison, we just need to argue that $b_n = n^2$ diverges to infinity.

Try $b_n = n$. Then we want $n^2 + n \leq 2n^3 + 1$. Although nothing cancels out, we can argue that, for all n, we have that $n^2 + n \leq n^2 + n^2 = 2n^2 \leq 2n^2 + 1 \leq 2n^3 + 1$. Of course that wasn't as simple of an argument as using $b_n = n^2$ though.

Another argument with $b_n = n$ is the following: $n^2 + n \leq n^3 + n^3 = 2n^3 < 2n^3 + 1$ since n^2 and n are both less than or equal to n^3. *(I think this is my favorite argument.)*

So you see that there are a variety of possible arguments.

We have shown that there are several choices of b_n that diverge to infinity and also satisfy $b_n \leq a_n$ for all n. Thus, by comparison Theorem 2.3.2, a_n also diverges to infinity.

Another possible solution is to note that $b_n = \frac{n^3}{n+1} < \frac{2n^3+1}{n+1}$ for all n and show that $\frac{1}{b_n}$ converges to zero and use Theorem 2.3.6, noting that $b_n > 0$ for all n as well. We have that $\frac{1}{b_n} = \frac{1}{n^2} + \frac{1}{n}$ which clearly converges to zero. Thus b_n diverges to $+\infty$ by Theorem 2.3.6 and a_n diverges to $+\infty$ by comparison.

4. (15 pts) Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence which converges to $A \in \mathbb{R}$ and $\{b_n\}_{n\in\mathbb{N}}$ a sequence which converges to $B \in \mathbb{R}$. Prove that $a_n + k \cdot b_n \to A + k \cdot B$ as $n \to \infty$ where $k \in \mathbb{R}$.

Solution:

Proof. Let $\epsilon > 0$ and choose n_1 such that $|a_n - A| < \frac{\epsilon}{2}$ for all $n \geq n_1$, and also choose n_2 such that $n \geq n_2$ implies that $|b_n - B| < \frac{\epsilon}{2|k|}$. Then for all $n \geq n^* = \max\{n_1, n_2\}$ we have that

$$\begin{aligned} |(a_n + kb_n) - (A + kB)| &= |(a_n - A) + k(b_n - B)| \\ &\leq |a_n - A| + |k| \cdot |b_n - B| \\ &< \frac{\epsilon}{2} + |k| \cdot \frac{\epsilon}{2|k|} \\ &= \epsilon \end{aligned}$$

Thus $a_n + k \cdot b_n \to A + k \cdot B$. ∎
