MATH 413 – Real Analysis I – Fall 2020 – HW 04

SOLUTION

1. Section 2.3, Exercise 3(a)

   Solution:

   Prove that $a_n = \frac{n^2+1}{n-2}$ diverges to infinity.

   This problem is a bit ill-posed since the domain of values for n is not specified and the sequence is undefined at $n = 2$, therefor this is not a sequence of real numbers! Let's just assume that the index starts at $n = 3$, i.e. that we are considering the sequence $\left\{\frac{n^2+1}{n-2}\right\}_{n=3}^{\infty}$.

   Note that $n < n + \frac{1}{n} = \frac{n^2+1}{n} < \frac{n^2+1}{n-2}$. So given any $M > 0$ let $n^* = \max\{M, 3\}$. For any $n \geq n^*$ we have $M = n^* \leq n < n + \frac{1}{n} = \frac{n^2+1}{n} < \frac{n^2+1}{n-2}$ thus $\frac{n^2+1}{n-2}$ diverges to infinity.

2. Section 2.4, Exercise 2. Also: give an example of a converging sequence that does attain its maximum. You do not need to prove your results, but give some argument.

   Solution:

   $a_n = 1 - \frac{1}{n}$ converges to one, but never "attains" a maximum value. Assume $M = \max\{a_n\}$. Then $M < 1$ since $a_n < 1$ for all n. But then there is some $\epsilon > 0$ such that $M + \epsilon < 1$, i.e. that $M < 1 - \epsilon < 1$. Since a_n converges to one, there are infinitely many terms in the interval $(1 - \epsilon, 1]$. So M cannot possibly be the max.
   Let $a_1 = 1$ and $a_n = 1 - \frac{1}{n}$ for all $n \geq 2$. Then $\max\{a_n\} = 1$ and $a_n = 1$ for $n = 1$. Note that we could set a_1 to any value at or above 1, and the sequence would still attain this value as its maximum. For another example, consider $b_n = \frac{1}{n}$, a decreasing sequence that attains its maximum at $a_1 = 1$ also. Every decreasing sequence will attain its maximum, and every increasing sequence will attain its minimum.

3. Section 2.4, Exercise 5

   Solution:

   Prove that for an eventually decreasing sequence a_n, there are two possibilities:
   (a) a_n is bounded below by M, in which case there exists $L \geq M$ such that $a_n \to L$.
   (b) a_n is unbounded, in which case $a_n \to -\infty$.

   Proof. (a) If a_n is bounded from below and is eventually decreasing, then it converges by the monotone convergence theorem. Let L be its limit. Now we must show that $M \leq L$. Assume the opposite, that $L < M$. Then there exists an $\epsilon > 0$ such that $L + \epsilon < M$. Since $a_n \to L$ and is eventually decreasing, we know that there is an n^* such that $L \leq a_{n+1} < a_n < L + \epsilon < M$ for all $n \geq n^*$. This is a contradiction since M is a lower bound on a_n, and we cannot have $a_n < M$. Thus we conclude that $M \leq L$.

   (b) If a_n is eventually decreasing and unbounded, then by definition of unboundedness, there are terms below M for any $M \in \mathbb{R}$. Let $M < 0$. Only consider the part of the sequence which are deceasing. This just means we discard a finite number of terms. The remaining terms must still have no lower bound. Then there exists some n^* such that $a_{n^*} < M$ (for this part of the sequence where it is decreasing). Since it is decreasing, $a_n \leq a_{n^*} < M$ for all $n \leq n^*$. Since we can find this cutoff n^* for any arbitrary $M < 0$, we conclude that this sequence is diverging to $-\infty$. ∎

4. Consider the sequence defined by $a_1 = 1$ and $a_{n+1} = \sqrt{a_n + 1} + 1$ for all $n \geq 1$. Prove that this sequence converges and find its limit. *(Hint: use monotone convergence. Induction may be helpful to show the sequence is bounded and increasing.)*

Solution:

Step 1: show it is monotone increasing

Note that $a_1 = 1 < a_2 = \sqrt{2} + 1$. So it looks like it might be increasing. Assume $a_{n-1} < a_n$. Now $a_{n+1} = \sqrt{a_n + 1} + 1 > \sqrt{a_{n-1} + 1} + 1 = a_n$. This shows that $a_n < a_{n+1}$ also. Since we have show that $a_1 < a_2$ and that $a_{n-1} < a_n$ implies that $a_n < a_{n+1}$, by induction we have that $a_n < a_{n+1}$ for all n. So this sequence is strictly increasing.

Step 2: show it is bounded from above

Note that $a_1 \leq 3$. Assume $a_n \leq 3$. Then $a_{n+1} = \sqrt{a_n + 1} + 1 \leq \sqrt{3 + 1} + 1 = 2 + 1 = 3$. Thus induction shows that $a_n \leq 3$ for all n. By the monotone convergence theorem, we now know that this sequence converges.

Step 3: find the limit

Let $a_n \to A$. Since $a_{n+1} = \sqrt{a_n + 1} + 1$, we know that shifting the sequence index by 1 does not change the convergence behavior, so that $a_n \to A$ means that $a_{n+1} \to A$. We also know that adding 1 to a sequence gives $a_n + 1 \to A + 1$ and that taking square root of a sequence doesn't affect things, i.e. that $\sqrt{a_n + 1} \to \sqrt{A + 1}$ (see Theorem 2.2.1(e)). So we can take the limit of both sides of the recursive formula to get $A = \sqrt{A + 1} + 1$. Now we solve for A. $A - 1 = \sqrt{A + 1} \Rightarrow (A - 1)^2 = A + 1 \Rightarrow A = 0, 3$. Since we know the sequence is increasing and $a_1 = 1$, we cannot have $A = 0$, thus $a_n \to 3$.

It is an instructive exercise to see what happens when you choose different values for a_1. You will find that for any $a_1 \geq -1$ the sequence converges to 3. When $a_1 < 3$ it is increasing, and it is decreasing when $a_1 > 3$. Clearly $a_1 = 3$ gives a constant sequence.

5. Section 2.5, Exercise 11

Solution:

Prove that if a_n and b_n are two Cauchy sequences, then so are $a_n + b_n$ and $a_n b_n$. Do not use Theorem 2.5.9.

Proof. We first show that $a_n + b_n$ is Cauchy. Given $\epsilon > 0$ choose $n_1, n_2 \in \mathbb{N}$ such that $n, m \geq n_1$ implies that $|a_n - a_m| < \frac{\epsilon}{2}$ and $n, m \geq n_2$ implies that $|b_n - b_m| < \frac{\epsilon}{2}$. Then we have that $n \geq \max\{n_1, n_2\}$ implies that $|(a_n + b_n) - (a_m + b_m)| \leq |a_n - a_m| + |b_n - b_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus $a_n + b_n$ is Cauchy.

Now we show that $a_n b_n$ is Cauchy. Since a_n and b_n are Cauchy, we know they are bounded by Theorem 2.5.8. Let M_a and M_b be bounds on these sequences, respectively, so that $|a_n| \leq M_a$ and $|b_n| \leq M_b$ for all n. Now choose $n_1, n_2 \in \mathbb{N}$ such that $n, m \geq n_1$ implies that $|a_n - a_m| < \frac{\epsilon}{2M_b}$ and $n, m \geq n_2$ implies that $|b_n - b_m| < \frac{\epsilon}{2M_a}$. Then we have that $n \geq \max\{n_1, n_2\}$ implies that $|a_n b_n - a_m b_m| = |a_n b_n - a_m b_n + a_m b_n - a_m b_m| \leq |b_n| \cdot |a_n - a_m| + |a_m| \cdot |b_n - b_m| < M_b \cdot \frac{\epsilon}{2M_b} + M_a \cdot \frac{\epsilon}{2M_a} = \epsilon$. Thus $a_n b_n$ is Cauchy. ■

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6. (a) Consider the sequence

$$a_n = \begin{cases} (-1)^n + 3 & \text{when } n \text{ is a multiple of } 5 \\ \frac{1}{n} & \text{otherwise} \end{cases}$$

Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

(b) Now consider the subsequence $b_n = a_{5n}$ for $n \in \mathbb{N}$. Find $\limsup_{n\to\infty} b_n$ and $\liminf_{n\to\infty} b_n$.

Solution:

(a) $\limsup_{n\to\infty} a_n = 4$ because no matter how large a cutoff n^* we pick, there will be n that are multiple of 5 beyond it giving us $a_n = 5$. Of course $a_n \leq 5$ for all n.
$\liminf_{n\to\infty} a_n = 0$ because, excluding n that are multiples of 5 there will be infinitely many large n values using the $\frac{1}{n}$ part of the definition that get us very close to zero, in fact, arbitrarily close. We have that $a_n > 0$ for all n, but given any $\epsilon > 0$ we have that $0 < a_n < \epsilon$ for infinitely many n so for any n^* we have that $\inf\{a_n \mid n \in \mathbb{N}, n \geq n^*\} = 0$.

(b) Now extracting the subsequence $b_n = a_{5n}$ for $n \in \mathbb{N}$ excludes the $\frac{1}{n}$ part of the definition of a_n. We get $\limsup_{n\to\infty} b_n = 4$ and $\liminf_{n\to\infty} b_n = 2$.

7. (OPTIONAL) Prove the following theorem about "swapping the order" of strictly increasing convergent sequences.

> **Theorem.** Let a_n and b_n be two strictly increasing sequences converging to the same limit and satisfying $a_n < b_n$ for all n. Prove that the exists a subsequence of a_n, $c_n = a_{f(n)}$ that satisfies $b_n < c_n$ for all n.

Solution:

Let $a_n \to A$ and $b_n \to A$. We know that $a_n < b_n < A$ for all n since they are strictly increasing sequences. Thus we have that $0 < A - b_n < A - a_n$ for all n.

Now we will create a subsequence of a_n. Since $0 < A - b_n$ for all n, for each fixed $k \in \mathbb{N}$ we can find a $n^* \in \mathbb{N}$ such that for all $n \geq n^*$ we have that $|a_n - A| < A - b_k$. Note that here n is ranging from n^* to infinity, and k is a fixed constant.

Also, since b_n is strictly increasing, we know that $A - b_n$ is strictly decreasing. This gives us that when we choose a cutoff n^* such that $n \geq n^*$ gives us $|a_n - A| < A - b_k$ we also have that $|a_n - A| < A - b_k < A - b_{k-1}$.

Let $n = 1$, then choose n_1 such that $A - a_{n_1} = |a_{n_1} - A| < A - b_1$ (there shouldn't be any problem with this given what was stated above). Then let $n = 2$ and choose $n_2 > n_1$ such that $A - a_{n_2} < A - b_2$. Recall that we know there is an n^* such that $n \geq n^*$ implies that $|a_n - A| < A - b_2$ so we just need to choose an n_2 that satisfies $n_2 \geq \max\{n^*, n_1 + 1\}$. So we are essentially creating a sequence of cut-off n^*'s $\{n_1^*, n_2^*, \ldots\}$ and picking $n_1 \geq n_1^*$, $n_2 \geq \max\{n_2^*, n_1 + 1\}$, $n_3 \geq \max\{n_3^*, n_2 + 1\}$, ..., $n_k \geq \max\{n_k^*, n_{k-1} + 1\}$. For each k, we know that such an n_k can be chosen since there are always infinitely many natural number left to choose from. This is a consequence of induction: We can do this to get an n_1, and our ability to do this to get n_k implies that we can do it to get such an n_{k+1} therefore we can create such a sequence $\{n_k\}_{k\in\mathbb{N}}$.

Note that $A - a_{n_k} < A - b_k$ for all $k \in \mathbb{N}$. In other words, rearranging gives $b_k < a_{n_k}$ for all k. Now we have our subsequence of a_n given by $f(k) = n_k$ (note that $c_k = a_{f(k)}$) that is strictly larger than b_n for all n, that is that $b_k < a_{n_k} = c_k$. Note that the use of k to denote the index here is of no consequence. Here k is ranging over the set of all natural numbers, so we can simply rewrite "$b_k < c_k$ for all $k \in \mathbb{N}$" as "$b_n < c_n$ for all $n \in \mathbb{N}$."