SOLUTION

1. Let $f(x) = \frac{10x+3}{5x-2}$. Prove that $\lim_{x\to\infty} f(x) = 2$. Use Definition 3.1.1 directly, but no other results after it from 3.1.

*Solution:*

We want $|f(x) - 2| < \epsilon$ when $x \geq M$. We have that $f(x) - 2 = \frac{10x+3-10x+4}{5x-2} = \frac{7}{5x-2} < \epsilon$ when $\frac{1}{5}\left(\frac{7}{\epsilon} + 2\right) < x$. This also requires $5x - 2 > 0$ in order to allow the calculation to not flip the inequality. Now we write our proof.

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*Proof.* Let $\epsilon > 0$, and choose any $M > \max\left\{\frac{2}{5}, \frac{1}{5}\left(\frac{7}{\epsilon} + 2\right)\right\}$. Then for all $x \geq M$ we have that

$$\begin{aligned}|f(x) - 2| &= \left|\frac{10x+3}{5x-2} - 2\right| \\ &= \frac{7}{5x-2} \\ &\leq \frac{7}{5M-2} \\ &\leq \frac{7}{5\left[\frac{1}{5}\left(\frac{7}{\epsilon}+2\right)\right] - 2} = \epsilon\end{aligned}$$

Thus we have shown that for any arbitrary distance $\epsilon$ we wish to be from 2, we can find a cutoff $x$ value $M$ beyond which $f(x)$ is within $\epsilon$ of 2, satisfying the definition of $\lim_{x\to\infty} f(x) = 2$. ■

2. Let $f(x) = \frac{x^3-2x^2+x+7}{2x-1}$. Prove that $\lim_{x\to\infty} f(x) = +\infty$. Use Definition 3.1.9 directly, but no other results after it from 3.1.

*Solution:*

Note that $f(x) \approx \frac{1}{2}x$ for $x$ large enough. This means that if we choose a slightly smaller coefficient, say, $\frac{1}{3}$, then we can show that $f(x) \geq \frac{1}{3}$ for large enough $x$ values.

Here is some scratchwork: $f(x) = \frac{x^3-2x^2+x+7}{2x-1} \geq \frac{1}{3}x$ if and only if $3(x^3-2x^2+x+7) \geq x(2x-1)$ (cross-multiplying). This simplifies to $3x^3-6x^2+3x+21 \geq 2x^2-x$ and then to $3x^3-8x^2+4x+21 \geq 0$. We can argue that $3x^3$ is the dominant term of the left side and so will overtake the $-8x^2$ term when $x$ is large enough. First note that in our cross multiplication, we required $2x-1>0$ to prevent flipping the inequality (i.e. $x > \frac{1}{2}$).

You can simply argue that $3x^3-8x^2+4x+21$ is a polynomial with even leading coefficient, and therefore its right wide tail goes upwards, but I will present a more thorough argument here.

Note that $4x+21 \geq 0$ when $x > \frac{1}{2}$, so that requirement (to prevent flipping the inequality when cross multiplying) takes care of that part.

Now note that $3x^3 > 8x^2$ when $x > \frac{8}{3}$. Note that $\frac{8}{3} > \frac{1}{2}$. So we can safely say that when $x > \frac{8}{3}$ we have $3x^3 - 8x^2 + (4x+21) > 8x^2 - 8x^2 + 0 = 0$.

We conclude that $f(x) = \frac{x^3-2x^2+x+7}{2x-1} > \frac{1}{3}x$ when $x > \frac{8}{3}$. Note that this is a stronger bound on $x$ than we need. In fact, this is true for all $x > 0$, and even true for some negative $x$ values as well. It doesn't matter, we just needed to make sure it was eventually true. It doesn't matter if our bound is the best possible bound.

Now we are ready to write the proof.

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*Proof.* Let $K > 0$ and choose any $M > \max\{\frac{8}{3}, 3K\}$. Then we have that for all $x \geq M$ it is true that

$$\begin{aligned} f(x) &= \frac{x^3-2x^2+x+7}{2x-1} \\ &> \frac{1}{3}x && \text{(since } x \geq M > \frac{8}{3}\text{, and above scratchwork)} \\ &\geq \frac{1}{3}M && \text{(since } x \geq M\text{)} \\ &> \frac{1}{3}(3K) && \text{(since } M > 3K\text{)} \\ &= K \end{aligned}$$

Therefore $f(x) \to \infty$ as $x \to \infty$. ■

3. Section 3.2, Exercise 1(a).

*Solution:*

Show that $\lim_{x \to 0} (x+1)^3 = 1$.

Note that $(x+1)^3 - 1 = ((x+1)-1)((x+1)^2 + (x+1) + 1)$ by sum/difference of cubes factoring formula. Thus putting on absolute value bars and simplifying we get

$$|(x+1)^3 - 1| = |x| \cdot |x^2 + 3x + 3|.$$

Now we will bound the factor on the right. Since $x$ is going to zero, we will require $|x| < 1$. This makes $|x^2 + 3x + 3| < 7$. So now given $\epsilon > 0$, we will choose a $\delta > 0$ that also satisfied $\delta < 1$ and $\delta < \frac{\epsilon}{7}$ so that $|x| < \delta < 1$ means $|x^2 + 3x + 3| < 7$. Now we write up our proof.

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*Proof.* Let $\epsilon > 0$ and choose $\delta > 0$ such that $0 < \delta < \min\{1, \frac{\epsilon}{7}\}$. THen for all $x$ such that $0 < |x| < \delta$ we have

$$\begin{aligned}
|(x+1)^3 - 1| &= |x| \cdot |x^2 + 3x + 3| \\
&< \delta \cdot (|x|^2 + 3|x| + 3) \\
&< \delta \cdot (\delta^2 + 3 \cdot \delta + 3) \\
&< \delta \cdot (1^2 + 3 \cdot 1 + 3) \\
&= \delta \cdot 7 \\
&= \frac{\epsilon}{7} \cdot 7 = \epsilon
\end{aligned}$$

Thus $(x+1)^3 \to 1$ as $x \to 0$. ∎

4. Section 3.2, Exercise 8.

<u>*Solution:*</u>

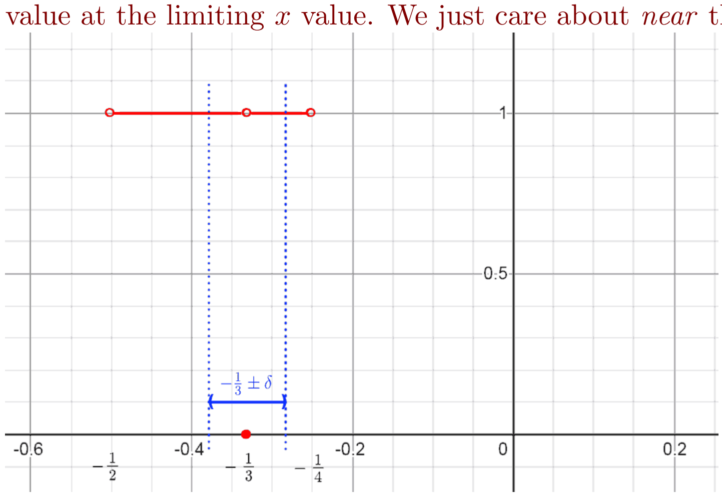
Consider the function $f : [-1, 1] \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 0 & \text{if } x = \pm\frac{1}{n} \text{ with } n \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

Find the given limits if possible and then prove that your results are correct.
(a) $\lim_{x\to\frac{3}{8}} f(x)$
(b) $\lim_{x\to-\frac{1}{3}} f(x)$
(c) $\lim_{x\to 0} f(x)$

(a) Notice that $f(\frac{3}{8}) = 1$ since $\frac{3}{8} \neq \frac{1}{n}$ for any $n \in \mathbb{N}$. Note that $\frac{1}{4} < \frac{3}{8} < \frac{1}{2}$ so that if $\delta < \frac{1}{8}$ and $|x - \frac{3}{8}| < \delta$ then $f(x) = 1$ also. So for any $\epsilon > 0$, let $\delta = \frac{1}{8}$, then $|f(x) - 1| < \epsilon$ when $0 < |x - \frac{3}{8}| < \delta = \frac{1}{8}$. Thus $\lim_{x\to\frac{3}{8}} f(x) = 1$.

(b) We have that $f(-\frac{1}{3}) = 0$ and that the nearest other points to $x = -\frac{1}{3}$ that have form $\pm\frac{1}{n}$ are $x = -\frac{1}{2}$ and $x = -\frac{1}{4}$. Notice that on the interval $(-\frac{1}{2}, -\frac{1}{4})$ that $-\frac{1}{3}$ is the only $x$ value in that interval where $f(x) = 0$ though. So if $0 < \delta < \frac{1}{12}$, then $|x - (-\frac{1}{3})| < \delta$ with $x \neq -\frac{1}{3}$ gives $f(x) = 1$ so that $|f(x) - 1| = 0$. Thus $\lim_{x\to-\frac{1}{3}} f(x) = 1$. See the picture below. It is important that when taking limits, we don't actually care about the function value at the limiting $x$ value. We just care about *near* that $x$ value.




(c) $\lim_{x\to 0} f(x)$ does not exist. For any $\delta > 0$ there will be $x$ values in the interval $(0, \delta)$ such that $f(x) = 0$ and some where $f(x) = 1$. Thus let $L \in \mathbb{R}$ be some candidate limiting value for $f$, let $\epsilon = \frac{1}{2}$, and let $\delta > 0$ be any positive number. Then there are $x_1, x_2 \in (0, \delta)$ such that $f(x_1) = 0$ and $f(x_2) = 1$. No matter what the value of $L$ is, either $|f(x_1) - L| \geq \frac{1}{2}$ or $|f(x_2) - L| \geq \frac{1}{2}$. Therefore $|f(x) - L| \geq \frac{1}{2}$ for some $x \in (0, \delta)$ no matter how small we set $\delta$.