MATH 413 – QUIZ 2 – Fall 2020 || 50 POINTS

SOLUTIONS

1. (10 pts) Let $a_n = (-1)^n + \frac{1}{n}$. Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$. You do not need to write a proof, but you should show clear work and reasoning to justify your answer. Find a convergent subsequence.

Solution:

limsup:

$$\begin{aligned}\limsup_{n\to\infty} a_n &= \lim_{n\to\infty} \sup\{a_k \mid k \geq n\} \\ &= \lim_{n\to\infty} \sup\left\{(-1)^k + \frac{1}{k} \mid k \geq n\right\} \\ &= \begin{cases} \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right) & \text{if } n \text{ even} \\ \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n+1}\right) & \text{if } n \text{ odd} \end{cases} \\ &= 1\end{aligned}$$

Since $\frac{1}{k}$ is decreasing, then the supremum is achieved at the first even k that is greater than or equal to n.

liminf:

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k \mid k \geq n\} = \lim_{n\to\infty} \sup\left\{(-1)^k + \frac{1}{k} \mid k \geq n\right\} = \lim_{n\to\infty}(-1+0) = -1$$

For any number $a > -1$, there will an odd $k \in \mathbb{N}$ such that $a_k = (-1)^k + \frac{1}{k} = -1 + \frac{1}{k} < a$ so a cannot be the infimum.

convergent subsequence:

If we let $b_n = a_{2n}$, then $b_n = (-1)^{2n} + \frac{1}{2n} = 1 + \frac{1}{2n}$ and this sequence converges to 1.

2. (20 pts) Let $f(x) = \frac{x^2+x-6}{5x+10}$. Evaluate $\lim_{x\to\infty} f(x)$. Prove your result directly using only Section 3.1 definitions.

Solution:

For x large, I suspect $f(x) \approx \frac{1}{5}x$ simply by considering the leading power of x on top and bottom of the rational function. So we should be able to find a constant C such that $f(x) \geq Cx$ is eventually true. This works if $C < \frac{1}{5}$. I will use $C = \frac{1}{10}$ since it simplifies things nicely.

$f(x) = \frac{x^2+x-6}{5x+10} \geq \frac{1}{10}x$ simplifies to $x \geq \sqrt{12}$.

Now we also want $\frac{1}{10}x > K$ which gives $x > 10K$.

So given any $K > 0$, choose $M > \max\{10K, \sqrt{12}\}$, then $x \geq M$ implies that

$$f(x) \geq \frac{1}{10}x \geq \frac{1}{10}M > \frac{1}{10}10K = K$$

3. (20 pts) Let $f(x) = \frac{x^2-4}{x-2}$. Evaluate $\lim_{x\to 2} f(x)$. Prove your result directly using only Section 3.2 definitions.

Solution:

Note that $f(x) = \frac{(x-2)(x+2)}{x-2} = x+2$ as long as $x \neq 2$. We suspect that $f(x) \to 4$ as $x \to 2$. So we wish to find a δ such that $|f(x) - 4| < \epsilon$ when $0 < |x-2| < \delta$. Note that $x \neq 2$ holds always since $2 \notin Dom(f)$, and $0 < |x-2| < \delta$ requires $x \neq 2$ anyways, so we are good to go.
$|f(x) - 4| = |x-2|$ so we can simply let $\delta = \epsilon$.

Proof. Let $\epsilon > 0$ and let $\delta = \epsilon$. Then for all $x \in Dom(f)$ such that $0 < |x-2| < \delta = \epsilon$ we have that

$$\begin{aligned} |f(x) - 4| &= \left|\frac{(x-2)(x+2)}{x-2} - 4\right| \\ &= |(x+2) - 4| \quad (\text{since } x \neq 2) \\ &= |x-2| \\ &< \delta \\ &= \epsilon \end{aligned}$$

Thus $f(x)$ goes to 4 as x goes to 2.