MATH 413 – Real Analysis I – Fall 2020 – HW 06

Please answer the questions below. I prefer to have your file scanned and saved as a single pdf and submitted to Blackboard.

Name the pdf file: `hw06_math413_lastname.pdf` with "lastname" of course replaced by your last name.

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1. Consider the following functions and determine where they are continuous. Prove your results using the definition of continuity.

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } 1 \leq x < 2 \\ 2 & \text{if } x > 2 \end{cases}$$

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 1 & \text{if } 1 \leq x < 2 \\ 2 & \text{if } x \geq 2 \end{cases}$$

2. Section 4.1, Exercise 5(a).

3. Section 4.1, Exercise 5(b) and (e). *(Hint: Consider the sequential definition of function limits. and look at the relevant results for limits of sequences.)*

4. (Extension of previous problem) If f is continuous at $x = a$ and $f(a) > 0$, prove that $[f(x)]^r$ is continuous at $x = a$ for $r \in \mathbb{R}$. *(Hint: Consider the sequential definition of function limits, and look at my supplemental notes on exponentiation of sequences.)*

5. Section 4.1, Exercise 6(a-e). You don't need to give full proofs, but you should show some valid reasoning.

6. (Optional, extra credit) Prove that the exponential function $f(x) = b^x$, with $b > 0$ a real constant, is continuous on $\mathbb{R}$. That is if $a \in \mathbb{R}$ prove that $\lim_{x \to a} b^x = b^a$. This is part of Section 4.1, Exercise 11(a). *(Hints: Try the case for $b > 1$ first and consider how b^a is defined for both rationals and irrationals. See my supplementary notes on exponentiation.)*

   Here are some steps that work:

   (a) Consider the case $b = 1$ which should be easy.

   (b) Show that $\lim_{x \to 0} b^x = b^0 = 1$ by using the sequential characterization of limits. If x_n is an arbitrary sequence converging to 0, then note that there is a subsequence of $\{x_n\}$ given by $y_n = x_{f(n)}$ such that $|y_n| < \frac{1}{n}$. What can you say about $b^{|y_n|}$?

   (c) Now show that $\lim_{x \to a} b^{x-a} = 1$. Thus it should give $b^{-a} \cdot \lim_{x \to a} b^x$. Do this by arguing that $\lim_{x \to a} x - a = 0$ and using the previous step.

   (d) Now consider the case $0 < b < 1$ by noting that $\frac{1}{b} > 1$ and apply the previous steps. Use known theorems about limits of functions, specifically limits of ratios of functions.