MATH 413 – EXAM 1 – Fall 2020 || 100 POINTS

SOLUTIONS

1. (10 pts) Answer the following **True (T) or False (F)** questions about sequences of real numbers.

$\boxed{T}$	F	If $\lim_{n\to\infty} a_n = +\infty$ and $a_n \neq 0$ for any n, then $\frac{1}{a_n} \to 0$. See Theorem 2.3.6
T	$\boxed{F}$	If $\{x_n\}$ converges and $\{y_n\}$ is bounded, then $\{x_n y_n\}$ converges. Consider $x_n = 1$ and $y_n = (-1)^n$
$\boxed{T}$	F	A sequence is convergent if and only if it is Cauchy. See Theorem 2.5.9
$\boxed{T}$	F	Every bounded sequence has a convergent subsequence. See Theorem 2.6.4
T	$\boxed{F}$	A sequence is convergent if and only if it is bounded. $(-1)^n$ is bounded but not convergent
$\boxed{T}$	F	If $\{x_n\}$ is increasing and bounded from above, it converges. See Theorem 2.4.4, Monotone convergence theorem
$\boxed{T}$	F	If $a_n \to A$ as $n \to \infty$, then every subsequence of $\{a_n\}$ also converges to A. See Theorem 2.6.5
$\boxed{T}$	F	Every Cauchy sequence is bounded. Cauchy $\Rightarrow$ convergent $\Rightarrow$ bounded
T	$\boxed{F}$	Every bounded sequence is Cauchy. $(-1)^n$ is bounded but not Cauchy
T	$\boxed{F}$	Every monotone sequence is bounded. $x_n = n$ is unbounded

2. (10 pts) Discuss in your own words what it means for a real number to "exist." You should write at least 2-3 sentences, but not like a long multi-page essay (unless if you *really* want to). As some thought prompts, consider the following (you don't need to answer these questions specifically though, they are just meant to help get your thought process going). We proved in class that $\sqrt{2} \in \mathbb{R}$. What does it mean that $\sqrt{2}$ is a real number? What does it even mean that 2 is a natural number? Does "2" exist?
(Hint: A "correct" answer can be achieved with a single short sentence. I just want to see your own honest individual thoughts here though and am not as concerned with a "correct" answer.)

Solution:

Existence in mathematics can be thought of as meaning that the object under consideration can be defined in a formal way that is sensible or that the object can be show to be a member of some set under consideration. So we say that x exists when we have formally constructed a set S (using axioms, theorems, and definitions that follow the rules of logic) and we can mathematically show that $x \in S$.

Example: We say $\sqrt{2}$ exists because we are able to prove that $\sqrt{2} \in \mathbb{R}$. The formal rules and axioms that we use to "create" $\mathbb{R}$ necessarily imply that we are also creating $\sqrt{2}$.

3. (20 pts) Prove directly using the definition of sequence convergence that $\left\{\dfrac{n^2+1}{n^2+n+2}\right\}_{n\in\mathbb{N}}$ converges.

Solution:

Given $\epsilon > 0$ choose any $n^* > \frac{1}{\epsilon}$. Then for any $n \geq n^*$ we have that

$$|a_n - 1| = \left|\frac{-n-1}{n^2+n+2}\right| = \frac{n+1}{n^2+n+2} < \frac{n+1}{n^2+n} = \frac{n+1}{n(n+1)} = \frac{1}{n} \leq \frac{1}{n^*} < \frac{1}{1/\epsilon} = \epsilon$$

Thus a_n converges to 1.

4. (20 pts) You are given that sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded. Prove that the sequence $\left\{\dfrac{a_n}{n}\right\}_{n\in\mathbb{N}}$ converges.

Solution:

Let $M > 0$ such that we have that $|a_n| \leq M$ for all n. Then $\left|\frac{a_n}{n}\right| \leq \frac{M}{n}$ for all n. Given $\epsilon > 0$ choose $n^* > \frac{M}{\epsilon}$. Then for any $n \geq n^*$ we have that $\left|\frac{a_n}{n}\right| \leq \frac{M}{n} \leq \frac{M}{n^*} < \frac{M}{M/\epsilon} = \epsilon$. Thus $\frac{a_n}{n}$ converges to zero.

5. (20 pts) Prove directly using the definition of limit for functions that $\lim_{x\to 5} f(x) = 1$.

$$f(x) = 2x - 9$$

Solution:

Given $\epsilon > 0$ choose $\delta = \frac{\epsilon}{2}$. Then for all x such that $0 < |x-5| < \delta$ we have $|f(x) - 1| = |2x - 10| = 2|x-5| < 2\delta = \epsilon$.

6. (20 pts) Let L be a positive real number. You are given that $\lim_{x\to\infty} f(x) = L$. Prove that $\lim_{x\to\infty} xf(x) = +\infty$. Directly use the definition of a function tending towards infinity.

Solution:

Since $L > 0$ we know that eventually $f(x) > 0$. Let $\epsilon_1 = \frac{L}{2}$ and choose M such that for all $x \geq M_1$ we have that $|f(x) - L| < \epsilon_1 = \frac{L}{2}$. Then when $x \geq M_1$ we have that $\frac{L}{2} < f(x) < \frac{3L}{2}$. Now given $K > 0$ choose $M_2 > \frac{2K}{L}$. Now let $M = \max\{M_1, M_2\}$.

Whenever $x \geq M$ we have that $x \geq M_2 > \frac{2K}{L}$ and $x \geq M_1$ which gives us that $\frac{L}{2} < f(x)$. Therefor $x \geq M$ implies that $xf(x) > \frac{2K}{L} \cdot \frac{L}{2} = K$. Thus $xf(x) \to \infty$ as $x \to \infty$.