MATH 413 – Real Analysis I – Fall 2020 – HW 06

SOLUTIONS

1. Consider the following functions and determine where they are continuous. Prove your results using the definition of continuity.

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } 1 \leq x < 2 \\ 2 & \text{if } x > 2 \end{cases}$$

Solution:

$x = 0$ is an isolated point, so f is continuous there. All x-values near $x = 1$ that are in the domain will give $f(x) = 1$, so f is clearly continuous there. There is also no problem at $x = 2$ since thata is not part of the domain. Thus f is a continuous function.

Let $\epsilon > 0$.

Case I: Given $a = 0$ and let $0 < \delta < 1$. Then $x \in (-\delta, \delta)$ in the domain implies $x = 0$ also. Thus $f(x) = f(0) = 0$ and hence $|f(x) - f(0)| = 0 < \epsilon$.

Case II: Given $a = 1$, let $0 < \delta < 1$. Then $x \in [1, 1 + \delta)$ gives $f(x) = 1$ and hence $|f(x) - f(1)| = 0 < \epsilon$.

Case III: Given a such that $1 < a < 2$, let $\delta = \min\{|x - 1|/2, |x - 2|/2\}$. Then $x \in (a - \delta, a + \delta)$ gives $f(x) = 1$ and hence $|f(x) - f(a)| = 0 < \epsilon$.

Case IV: Given a such that $a > 2$, let $\delta = |x - 2|/2$. Then $x \in (a - \delta, a + \delta)$ gives $f(x) = 2$ and hence $|f(x) - f(a)| = 0 < \epsilon$.

Thus f is continuous at every point of its domain and so is a continuous function.

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 1 & \text{if } 1 \leq x < 2 \\ 2 & \text{if } x \geq 2 \end{cases}$$

Solution:

We only need to consider what happens at $x = 1$ and at $x = 2$ since we know that $h(x) = x$ is a continuous polynomial, and that $g(x) = f(x)$ above when $x \geq 1$ except for $x = 2$.

Case I: Given $a = 1$, let $\delta = \min\{1, \epsilon\}$. Then $x \in (1 - \delta, 1 + \delta)$ gives $f(x) = 1$ or $f(x) = x$. If $x < 1$, then $|f(x) - f(1)| = |x - 1| < \delta \leq \epsilon$. If $x \geq 1$, then $|f(x) - f(1)| = 0 < \epsilon$. So f is continuous at $x = 1$.

Case II: Let $\epsilon < 1$ be an arbitrary positive number. Given $a = 2$, let $\delta > 0$. Then $x \in (2 - \delta, 2 + \delta)$ in the domain will always contain some $x < 2$ with $f(x) = 1$ satisfying $|f(x) - f(2)| = |1 - 2| = 1 > \epsilon$. Thus f is not continuous at $x = 2$.

Therefore g is **not** a continuous function.

2. Section 4.1, Exercise 5(a).

   Solution:

   We know that $\lim_{x \to a} f(x) = f(a)$ thus $\lim_{x \to a} |f(x)| = |f(a)|$ by Theorem 3.2.5.

   Alternatively, note that if $|f(x) - f(a)| < \epsilon$, then $f(a) - \epsilon < f(x) < f(a) + \epsilon$. Assume that $f(a) > 0$ and choose $\delta > 0$ small enough so that $f(a)/2 < f(x) < 3f(a)/2$. Thus for $|x - a| < \delta$, $f(x)$ must also be positive, hence $||f(x)| - |f(a)|| = |f(x) - f(a)| < \epsilon$.

   A similar argument works for $f(a) < 0$ so that $3f(a)/2 < f(x) < f(a)/2$ eventually.

   If $f(a) = 0$, then eventually $|f(x)| < \epsilon$.

3. Section 4.1, Exercise 5(b) and (e). *(Hint: Consider the sequential definition of function limits. and look at the relevant results for limits of sequences.)*

   Solution:

   We know that $\lim_{x \to a} f(x) = f(a)$ thus $\lim_{x \to a} \sqrt{f(x)} = \sqrt{f(a)}$ and $\lim_{x \to a} [f(x)]^n = [f(a)]^n$ by Theorem 3.2.5.

   Alternatively, let x_n be an arbitrary sequence in the domain of f that converges to a. Then $f(x_n)$ is a sequence that converges to $f(a)$ since f is continuous at a. Now, by properties of convergent sequences, Theorem 2.2.1 gives us that $\sqrt{f(x_n)}$ converges to $\sqrt{f(a)}$ and $[f(x_n)]^k$ converges to $[f(a)]^k$ (note the use of k here as a fixed arbitrary natural number power so as to not confuse it with the sequence index).

4. (Extension of previous problem) If f is continuous at $x = a$ and $f(a) > 0$, prove that $[f(x)]^r$ is continuous at $x = a$ for $r \in \mathbb{R}$. *(Hint: Consider the sequential definition of function limits, and look at my supplemental notes on exponentiation of sequences.)*

   Solution:

   Note that it is important that $f(x) > 0$ for all $x \in Dom(f)$ otherwise $[f(x)]^r$ may not be defined for some x. I should have stated that in the problem.

   Let x_n be an arbitrary sequence in the domain of f that converges to a. Then $f(x_n)$ is a sequence that converges to $f(a)$ since f is continuous at a.

   See my supplemental notes on rational and irrational exponentiation of sequences. From those notes, we get that $[f(x_n)]^r$ converges to $[f(a)]^r$. Thus $[f(x)]^r$ is continuous at $x = a$.

5. Section 4.1, Exercise 6(a-e). You don't need to give full proofs, but you should show some valid reasoning.

Solution:

(a) f bounded on $[a, b]$ implies that f is continuous on $[a, b]$.

False. Counterexample: $f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & 1 \leq x \leq 2 \end{cases}$ is bounded but not continuous on $[0, 2]$.

(b) f continuous on (a, b) implies that f is bounded on (a, b).

False. Counterexample: $f(x) = \frac{1}{x}$ on $(0, 1)$ is continuous but unbounded.

(c) $[f(x)]^2$ continuous on (a, b) implies that f is continuous on (a, b).

False. Counterexample: $f(x) = \begin{cases} -1 & 0 < x < 1 \\ 1 & 1 \leq x < 2 \end{cases}$
$[f(x)]^2 = 1$ on $(0, 2)$ is continuous, but f is not.

(d) f and g not continuous on (a, b) implies that fg is not continuous on (a, b).

False. Counterexample: Consider f above and let $g = f$ so that f^2 is continuous. Of course, this is somewhat trivial, but other more complicated examples could be constructed too.

(e) f and g not continuous on (a, b) implies that $f + g$ is not continuous on (a, b).

False. Counterexample: take f as above, but let $g = -f$, then $(f + g)(x) = 0$ on $(0, 2)$.

6. (Optional, extra credit) Prove that the exponential function $f(x) = b^x$, with $b > 0$ a real constant, is continuous on $\mathbb{R}$. That is if $a \in \mathbb{R}$ prove that $\lim_{x\to a} b^x = b^a$. This is part of Section 4.1, Exercise 11(a). *(Hints: Try the case for $b > 1$ first and consider how b^a is defined for both rationals and irrationals. See my supplementary notes on exponentiation.)*

Here are some steps that work:

(a) Consider the case $b = 1$ which should be easy.

(b) Show that $\lim_{x\to 0} b^x = b^0 = 1$ by using the sequential characterization of limits. If x_n is an arbitrary sequence converging to 0, then note that there is a subsequence of $\{x_n\}$ given by $y_n = x_{f(n)}$ such that $|y_n| < \frac{1}{n}$. What can you say about $b^{|y_n|}$?

(c) Now show that $\lim_{x\to a} b^{x-a} = 1$. Thus it should give $b^{-a} \cdot \lim_{x\to a} b^x$. Do this by arguing that $\lim_{x\to a} x - a = 0$ and using the previous step.

(d) Now consider the case $0 < b < 1$ by noting that $\frac{1}{b} > 1$ and apply the previous steps. Use known theorems about limits of functions, specifically limits of ratios of functions.

Solution:

If $b = 1$, then $b^x = 1$ for all $x \in \mathbb{R}$, and is thus continuous.

We'll only consider $b > 1$ since if $0 < b < 1$, then $b^x = \frac{1}{c^x}$ for some $c > 1$ thus continuity of c^x implies continuity of $b^x = \frac{1}{c^x}$ by Theorem 4.1.8(c).

Let $b > 1$, and we will show that $\lim_{x\to 0} b^x = b^0 = 1$. Consider a sequence x_n which converges to zero. Then there is a subsequence $|x_{n_k}| \leq \frac{1}{k}$ for all $k \in \mathbb{N}$. We then have that $0 < b^{x_{n_k}} \leq b^{1/k}$. We know that $b^{1/k}$ converges to 1 as $k \to \infty$ (see the 3rd theorem in my supplemental notes on exponentiation of sequences). Thus by a squeeze, $b^{x_{n_k}}$ converges to 1 as $k \to \infty$. This implies that b^{x_n} converges to 1 as $n \to \infty$. The sequence x_n can be thought of as being composed of two subsequences, one, purely negative, and one purely positive. If there are infinitely-many positive and negative terms, then the above argument works on them individually and thus on x_n overall. This is a bit of a hand-wavey explanation, but if you want more detail, just ask me.

Now let $a \in \mathbb{R}$. It is clear that $\lim_{x\to a} b^{x-a} = 1$ since $x - a \to 0$ as $x \to a$. But we also must have that $\lim_{x\to a} b^{x-a} = \lim_{x\to a} b^x \cdot \lim_{x\to a} b^{-a}$. Thus $1 = \lim_{x\to a} b^x \cdot b^{-a}$ giving us that $\lim_{x\to a} b^x = b^a$. This shows that b^x is continuous at $x = a$.