
Please answer the questions below. I prefer to have your file scanned and saved as a single pdf and submitted to Blackboard.

Name the pdf file: `hw07_math413_lastname.pdf` with "lastname" of course replaced by your last name.

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1. Section 4.4 Exercise 1(d). *(Hint: First prove that $\sqrt[3]{x} - \sqrt[3]{t} \leq \sqrt[3]{x-t}$ for $x > t \geq 0$. Then prove that $f(x) = \sqrt[3]{x}$ is uniformly continuous on $[0, \infty)$ and $(-\infty, 0]$. Then since it is continuous at 0 you get that it is uniformly continuous on $(-\infty, 0] \cup [0, \infty) = \mathbb{R}$. There are other ways to solve this problem too, this is just a suggested route.)*

   Solution:

   Let $0 \leq t < x$. Then $\sqrt[3]{x} \leq \sqrt[3]{x-t} + \sqrt[3]{t}$ can be seen by cubing both sides. This gives $\sqrt[3]{x} - \sqrt[3]{t} \leq \sqrt[3]{x-t}$ and hence $|\sqrt[3]{x} - \sqrt[3]{t}| \leq \sqrt[3]{|x-t|}$.

   Let $t < x \leq 0$. Then $|\sqrt[3]{x} - \sqrt[3]{t}| = \sqrt[3]{x} - \sqrt[3]{t} = -\sqrt[3]{|x|} + \sqrt[3]{|t|}$. Since $0 \leq |x| < |t|$ and the above, we have that $-\sqrt[3]{|x|} + \sqrt[3]{|t|} \leq \sqrt[3]{|t| - |x|} = \sqrt[3]{-t + x} = \sqrt[3]{|x-t|}$. Thus $|\sqrt[3]{x} - \sqrt[3]{t}| \leq \sqrt[3]{|x-t|}$ also holds.

   Let $\epsilon > 0$ and $\delta = \epsilon^3$. Then $|x - t| < \delta$ and either $x, t \in [0, \infty)$ or $x, t \in (-\infty, 0]$ implies that $|\sqrt[3]{x} - \sqrt[3]{t}| \leq \sqrt[3]{|x-t|} < \sqrt[3]{\delta} = \epsilon$. Thus $f(x) = \sqrt[3]{x}$ is uniformly continuous on $[0, \infty)$ and $(-\infty, 0]$.

   Since it is also continuous at $x = 0$, by the theorem I gave you in class, it is uniformly continuous on $(-\infty, 0] \cup [0, \infty) = \mathbb{R}$.

   Alternative argument: Let $\epsilon > 0$ and $\delta = \frac{1}{8}\epsilon^3$. Then $|x - t| < \delta$ implies $x, t \in (-\infty, 0]$, $x, t \in [0, \infty)$, or $x, t \in (-\frac{1}{8}\epsilon^3, \frac{1}{8}\epsilon^3)$. In the first two cases, we get $|\sqrt[3]{x} - \sqrt[3]{t}| \leq \sqrt[3]{|x-t|} < \sqrt[3]{\delta} = \epsilon$ as before, and in the last case we get $|\sqrt[3]{x} - \sqrt[3]{t}| \leq |\sqrt[3]{x}| + |\sqrt[3]{t}| < |\sqrt[3]{\frac{1}{8}\epsilon^3}| + |\sqrt[3]{\frac{1}{8}\epsilon^3}| = \epsilon$. This way we don't need to use the theorem I gave you in class.

2. Show that $f(x) = x^2$ is uniformly continuous on $[0, 5)$. Directly us the definition of uniform continuity.

   Solution:

   Let $\epsilon > 0$. We have, since $x, t < 5$ that $|x^2 - t^2| = |x - t| \cdot |x + t| \leq 10|x - t|$ so let $\delta = \frac{\epsilon}{10}$. Then $|x - t| < \delta$ implies $|f(x) - f(t)| < \epsilon$.

3. Show that $f(x) = \frac{1}{x-1}$ is not uniformly continuous on $(1, \infty)$. *(Hint: Find sequences x_n and t_n that work with Remark 4.4.4.)*

   Solution:

   Let $x_n = 1 + \frac{1}{n}$ and $t_n = 1 + \frac{2}{n}$. Then $|x_n - t_n| = \frac{1}{n}$ and $|f(x_n) - f(t_n)| = |\frac{n}{1} - \frac{n}{2}| = \frac{n}{2} > \epsilon$ for n sufficiently large no matter what we fix $\epsilon > 0$ as.

4. (optional, bonus) Show that $f(x) = e^x$ is not uniformly continuous on $\mathbb{R}$. *(Hint: Consult my supplemental notes on the natural exponential. Argue that* $e^n > 2^n \geq n^2$ *for* $n \in \mathbb{N}$. *Then use* $t_n = n$ *and* $x_n = n + \frac{1}{n}$ *and show that* $f(x_n) - f(t_n) = e^n(e^{\frac{1}{n}} - 1)$. *Then consult my supplemental notes on the natural exponential Theorem 6 that shows* $(1 + \frac{1}{n})^n$ *is increasing and argue that* $e^{\frac{1}{n}} \geq 1 + \frac{1}{n}$. *Finally, put this all together to get that* f *is not uniformly continuous using this sequential characterization.)*

Solution:

Let $x_n = n + \frac{1}{n}$ and $t_n = n$ so that $|x_n - t_n| = \frac{1}{n}$. Now $|f(x_n) - f(t_n)| = |e^{n+\frac{1}{n}} - e^n| = e^n(e^{\frac{1}{n}} - 1)$. Note that $e^{1/n} > 1$, so we are allowed to remove the absolute value bars. Since $(1 + \frac{1}{n})^n$ is increasing and converges to e, we have that $e^{\frac{1}{n}} \geq 1 + \frac{1}{n}$. So we put this together to get that $|f(x_n) - f(t_n)| > \frac{e^n}{n}$. Now we also know that $e > 2$ thus $e^n > 2^n \geq n^2$ for all n. Finally this gives us that $|f(x_n) - f(t_n)| > \frac{e^n}{n} > \frac{n^2}{n} = n > \epsilon$ eventually.