SOLUTIONS

1. Section 4.4, Exercise 5, part (a, b, & d). Directly use the definition of uniform continuity.

   Solution:

   If functions f and g are uniformly continuous on their common domain D, prove that
   (a) $f \pm g$ is uniformly continuous on D.
   (b) cf is uniformly continuous on D for any real constant c.
   (d) fg is uniformly continuous on D if f and g are both bounded on D.

   (a) Let $\epsilon > 0$. Choose $\delta_1 > 0$ such that $|x - t| < \delta_1$ implies $|f(x) - f(t)| < \frac{\epsilon}{2}$, and choose $\delta_2 > 0$ such that $|x - t| < \delta_2$ implies $|g(x) - g(t)| < \frac{\epsilon}{2}$. Then let $\delta = \min\{\delta_1, \delta_2\}$ so that whenever $|x - t| < \delta$ we have that

$$|(f(x) \pm g(x)) - (f(t) \pm g(t))| \leq |f(x) - f(t)| + |g(x) - g(t)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

   (b) Assume $c \neq 0$, and let $\epsilon > 0$. Let $\delta > 0$ such that $|x - t| < \delta$ implies $|f(x) - f(t)| < \frac{\epsilon}{|c|}$. Then it follows that $|cf(x) - cf(t)| < \epsilon$.

   (c) Assume $|f(x)| \leq M_f$ and $|g(x)| \leq M_g$ with M_f, M_g being two constants. Also let $M = \max\{M_f, M_g\}$. Then let $\delta > 0$ such that $|x - t| < \delta$ implies that both $|f(x) - f(t)| < \frac{\epsilon}{2M}$ and $|g(x) - g(t)| < \frac{\epsilon}{2M}$. Now we have that whenever $|x - t| < \delta$ we get

$$\begin{aligned} |f(x)g(x) - f(t)g(t)| &= |f(x)g(x) - f(t)g(x) + f(t)g(x) - f(t)g(t)| \\ &\leq |g(x)| \cdot |f(x) - f(t)| + |f(t)| \cdot |g(x) - g(t)| \\ &\leq M_g \cdot |f(x) - f(t)| + M_f \cdot |g(x) - g(t)| \\ &< M_g \cdot \frac{\epsilon}{2M} + M_f \cdot \frac{\epsilon}{2M} \\ &< \epsilon \end{aligned}$$

2. Section 5.1 Exercise 3(b,e).

Solution:

(b) $f(x) = \begin{cases} x^2+1 & x<1 \\ 2x & x \geq 1 \end{cases}$

We can see that $f(1-) = f(1)$ thus f is continuous at $x=1$. Let's check differentiability directly:

$$\begin{aligned} \frac{f(1+h)-f(1)}{h} &= \begin{cases} \frac{2(1+h)-2}{h} & h>0 \\ \frac{(1+h)^2+1-2}{h} & h<0 \end{cases} \\ &= \begin{cases} 2 & h>0 \\ 2+h & h<0 \end{cases} \\ &\xrightarrow{h\to 0} 2 \end{aligned}$$

Thus $f'(1) = 2$ does indeed exist.

(e) $f(x) = \begin{cases} x & \text{for } x \text{ rational} \\ 0 & \text{for } x \text{ irrational} \end{cases}$

$$\begin{aligned} \frac{f(h)-f(0)}{h} &= \begin{cases} \frac{h-0}{h} & \text{for } h \text{ rational} \\ \frac{0-0}{h} & \text{for } h \text{ irrational} \end{cases} \\ &= \begin{cases} 1 & \text{for } h \text{ rational} \\ 0 & \text{for } h \text{ irrational} \end{cases} \end{aligned}$$

Thus $f'(0)$ does not exist!

As an additional exercise, you should show that the function in (d) is actually differentiable at $x=0$.

3. Section 5.1 Exercise 15.

Solution:

If the function $xf(x)$ has a derivative at $x = a \neq 0$, prove that f is differentiable at $x = a$.

We know that $\lim_{h\to 0} \frac{(a+h)f(a+h)-af(a)}{h}$ exists.

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{(a+h)f(a+h)-(a+h)f(a)}{(a+h)h} \\
&= \frac{(a+h)f(a+h)-af(a)-hf(a)}{(a+h)h} \\
&= \frac{(a+h)f(a+h)-af(a)}{(a+h)h} - \frac{f(a)}{a+h} \\
&\xrightarrow{h\to 0} \frac{1}{a}\cdot\left(\frac{d}{dx}[xf(x)]\Big|_{x=a}\right) - \frac{f(a)}{a}
\end{aligned}$$

Thus if the derivative of $xf(x)$ at $x = a$ is L, that is, that $\frac{d}{dx}[xf(x)]\big|_{x=a} = L$, then $f'(a) = \frac{L}{a} - \frac{f(a)}{a}$.

4. Section 5.2, Exercise 3:

Solution:

If function f is differentiable and c is a real constant, prove that $(cf)'(x) = cf'(x)$ without using Theorem 5.2.1.

$$\lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = c\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = c\cdot f'(x)$$