MATH 413 – (simulated) QUIZ 4 – Fall 2020

SOLUTIONS

1. Find the pointwise limit of the sequence of functions $f_n(x) = \sqrt[n]{x}$. Prove that the convergence is not uniform.

Solution:

For $0 < x$, we have that $\sqrt[n]{x} \to 1$ as $n \to \infty$, and at $x = 0$ we have $\sqrt[n]{0} = 0$.
So the pointwise limit is

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & 0 < x \end{cases}$$

Let $\epsilon = \frac{1}{4}$. Consider the sequence of x-values $x_n = \frac{1}{2^n}$. We know that the pointwise limit at each of these points, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$, is always one. That is, that, $f_n(x_k) \to 1$ as $n \to \infty$ for any fixed $k \in \mathbb{N}$.

We have that $f_n(x_n) = \frac{1}{2}$. So we have constructed a sequence of x-values that will always be exactly 1/2 distance from their individual pointwise limits of one. By Remark 8.2.5(c), this is sufficient to show that the convergence is not uniform. I.e. $|f_n(x_n) - 0| = \frac{1}{2} > \epsilon = \frac{1}{4}$ thus the convergence cannot be uniform.

2. Prove that the following sequence of functions is converges uniformly to $f(x) = 0$ on $[0, \infty)$.

$$f_n(x) = nxe^{-n^2 x}$$

Solution:

The pointwise limit for all x is $f(x) = 0$.

We first show that there is an absolute maximum.
$f_n'(x) = n(1 - xn^2)e^{-n^2 x}$
$f_n''(x) = n^3(xn^2 - 2)e^{-n^2 x}$
Thus we have a local max at $x = \frac{1}{n^2}$. Therefore $f_n(x) = nxe^{-n^2 x} \leq \frac{1}{ne} < \frac{1}{n}$. We can see that f_n is an increasing function to the left of the max and a decreasing function to the right of the max. Furthermore $f_n(x)$ goes to zero as x goes to infinity. Thus all we have to consider is this maximum function value and see if we can bound it below an arbitrary ϵ.

Let $\epsilon > 0$ and pick any $n^* > \frac{1}{\epsilon}$. Then for all $n \geq n^*$ we have that $|f_n(x) - 0| = nxe^{-n^2 x} \leq \frac{1}{ne} < \frac{1}{n} \leq \frac{1}{n^*} < \epsilon$ for all $x \in [0, \infty)$. Thus f_n converges uniformly to the zero function.