MATH 413 – QUIZ 3 – Fall 2020 || 50 POINTS

SOLUTIONS

1. (20 pts) Prove that the following function is continuous at $x = 0$. Directly use the definition of continuity.

$$f(x) = \begin{cases} 2x & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & \text{for } x \in \mathbb{Q} \end{cases}$$

Solution:

Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{2}$. Then when $|x| < \delta$ we have that $f(x) = 0 < \epsilon$ is x is rational and $|f(x)| = 2|x| < 2\delta = \epsilon$ when x is irrational. Thus in either case we get that whenever $|x - 0| < \delta$ implies $|f(x) - f(0)| = |f(x)| < \epsilon$ showing that f is indeed continuous at $x = 0$.

Note that f is discontinuous at every other real number.

2. (30 pts) Prove that $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[2, 5]$. Directly use the definition of uniform continuity.

Solution:

Let $\epsilon > 0$. We wish to find a $\delta > 0$ so that $|x - t| < \delta$ implies that $|f(x) - f(t)| < \epsilon$.

Notice that:

$$|f(x) - f(t)| = \left| \frac{t^2 - x^2}{x^2 t^2} \right| = |x - t| \cdot |x + t| \cdot \frac{1}{x^2 t^2} < |x - t| \cdot 10 \cdot \frac{1}{16}$$

So we can choose $\delta = \frac{8}{5}\epsilon$.

Now when $|x - t| < \delta = \frac{8}{5}\epsilon$ we get that

$$|f(x) - f(t)| = |x - t| \cdot |x + t| \cdot \frac{1}{x^2 t^2} < \delta \cdot \frac{5}{8} = \frac{8}{5}\epsilon \cdot \frac{5}{8} = \epsilon$$

showing that f is indeed uniformly continuous on the given domain.

If the left-hand endpoint of the domain extended to zero f would not be uniformly continuous though, i.e. on $(0, a]$ for any positive a.

Edit: I am seeing many use the fact that $\frac{x+t}{x^2t^2}$ is bounded by $\frac{1}{4}$ at $x = t = 2$. This is correct, but it needs to be justified carefully. The bound I used in my solution above is "conservative" and doesn't require much justification. Here is a way to justify the $\frac{1}{4}$ bound; it uses partial derivatives.

Consider $f(x, t) = \frac{x+t}{x^2t^2}$. The partial derivative in x is $\frac{\partial f}{\partial x} = -\frac{1}{x^2t^2} - \frac{2}{x^3 t} < 0$ and thus this function (surface) is decreasing in the positive x-direction. Taking $\frac{\partial f}{\partial t}$ we can see that it is also decreasing in the positive t-direction. Thus on the limited domain we are considering, the maximum does indeed occur at $x = t = 2$.