MATH 413 – Real Analysis I – Fall 2020 – HW 09

SOLUTIONS

1. Section 8.1, Exercise 1, part (d).

   Solution:

   $f_n(x) = \frac{1}{n}e^{-n^2x^2}$ on $[0, \infty)$

   The pointwise limit is $f(x) = 0$ for all x. Notice that $0 < e^{-n^2x^2} \leq 1$ for all $x \in [0, \infty)$ and all $n \in \mathbb{N}$. Thus we have $0 < f_n(x) \leq \frac{1}{n}$.

   In fact this sequence also converges uniformly.

2. Section 8.1, Exercise 3.

   Solution:

   Suppose that $f_n, g_n : D \to \mathbb{R}$ and converge pointwise to f and g respectively. Prove that $f_n \pm g_n$ converges pointwise to $f \pm g$.

   Let $\epsilon > 0$ and let $x_0 \in D$ be an arbitrary point in the domain. Choose n_1^* so that $n \geq n_1^*$ implies $|f_n(x_0) - f(x_0)| < \frac{\epsilon}{2}$. We know we can do this at each x_0 since we assume pointwise convergence. Also choose n_2^* so that $n \geq n_2^*$ implies $|g_n(x_0) - g(x_0)| < \frac{\epsilon}{2}$. Then $\big|(f_n(x_0) + g_n(x_0)) - (f(x_0) + g(x_0))\big| \leq |f_n(x_0) - f(x_0)| + |g_n(x_0) - g(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. Thus $f_n + g_n$ converges pointwise to $f + g$ at x_0.

   The proof for $f_n - g_n$ is nearly identical.

3. Prove that $f_n(x) = \frac{x^n}{n}$ converges uniformly to $f(x) = 0$ on $[0, 1]$.

   Solution:

   Let $\epsilon > 0$ and choose $n^* > \frac{1}{\epsilon}$. Then for any $x \in [0, 1]$ and any $n \geq n^*$ we have $|f_n(x) - 0| = \frac{x^n}{n} \leq \frac{1}{n} \leq \frac{1}{n^*} < \epsilon$. Thus f_n converges uniformly to $f(x) = 0$ on $[0, 1]$.

   It is important to realize that the convergence is uniform since the argument does not depend on having fixed x to be any particular value. It works well in this case since x is bounded. Of course uniform convergence does not depend in general on x being bounded. Convergence can be uniform on unbounded domains as well.

4. Section 8.2, Exercise 5.

<u>*Solution:*</u>

If f_n and g_n are sequences of bounded functions that converge uniformly on D to functions f and g, respectively, prove that the sequence $f_n g_n$ converges uniformly to fg on D.

We will use the following results:

Definition 8.1.13. A sequence f_n is uniformly bounded on D if and only if there exists a real constant K such that $|f_n(x)| \leq K$ for all $x \in D$ and $n \in \mathbb{N}$.

Section 8.2, Exercise 3. If f_n is a sequence of bounded functions that converges uniformly to f on D, prove that

(a) f is bounded. (Note that this need not be true if convergence is not uniform. See Exercise 2(d) of Section 8.1.)

(b) f_n is uniformly bounded.

Proof of Section 8.2, Exercise 3(a).

Let $\epsilon = 1$. Then we know there is an n^* such that $n \geq n^*$ implies $|f_n(x) - f(x)| < 1$ regardless of $x \in D$. In particular $|f_{n^*}(x) - f(x)| < 1$ which implies in turn by rearranging that $f_{n^*}(x) - 1 < f(x) < f_{n^*}(x) + 1$ for all $x \in D$. Since we know that each f_n is bounded, then for n^* we know there is an $M > 0$ such that $-M \leq f_{n^*}(x) \leq M$ for all $x \in D$. This gives us that $-M - 1 < f(x) < M + 1$ i.e. that f is bounded by $M + 1$.

Note that we only know that each f_n is bounded. We **do not** know that there is a "global constant" that bounds all f_n simultaneously yet. That is what the next part is for.

Proof of Section 8.2, Exercise 3(b).

By the above, we have that f is bounded by $M+1$ and that $|f_n(x) - f(x)| < 1$ for all $n \geq n^*$. Rearranging this gives us that $f(x) - 1 < f_n(x) < f(x) + 1$ for all $n \geq n^*$. Putting in the bound on f gives $-M - 1 - 1 \leq f(x) - 1 < f_n(x) < f(x) + 1 \leq M + 1 + 1$ for all $n \geq n^*$, i.e. that $|f_n(x)| \leq M + 2$ for all $n \geq n^*$.

Then let M_k be the bound for f_k, for $k = 1, 2, \dots, n^* - 1$. Since it was assumed that the seuqence f_n is bounded, we know that each function in the sequence has a bound, thus we can define all of these M_k's. Now take $K = \max\{M_1, M_2, \dots, M_{n^*-1}, M + 2\}$. We have that K is a bound for all f_n, i.e. that $|f_n(x)| \leq K$ for all $n \in \mathbb{N}$ and $x \in D$. Thus K is a uniform bound and the sequence f_n is uniformly bounded.

Proof of Section 8.2, Exercise 5. Now on to the actual homework problem!

Since f_n and g_n are bounded sequences and converge uniformly to f and g, respectively, then we know that f_n and g_n are uniformly bounded and that f and g are bounded (by the above results just presented). Let K_f and K_g be the uniform bounds for f_n and g_n and M_f and M_g be the bounds for f and g, i.e. that

$$|f_n(x)| \leq K_f \text{ and } |g_n(x)| \leq K_g \text{ for all } n \in \mathbb{N} \text{ and } x \in D$$

$$|f(x)| \leq M_f \text{ and } |g(x)| \leq M_g \text{ for all } x \in D$$

Now let $\epsilon > 0$. Pick n_1^* such that $n \geq n^*$ implies that $|f_n(x) - f(x)| < \frac{\epsilon}{2M_g}$, and pick n_2^* such that $n \geq n^*$ implies that $|g_n(x) - g(x)| < \frac{\epsilon}{2K_f}$, both holding true for all $x \in D$. Then for all $x \in D$ we have that

$$\begin{aligned}
|f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \\
&\leq |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)| \\
&\leq |f_n(x)| \cdot |g_n(x) - g(x)| + |f_n(x) - f(x)| \cdot |g(x)| \\
&\leq K_f \cdot |g_n(x) - g(x)| + |f_n(x) - f(x)| \cdot M_g \\
&< K_f \cdot \frac{\epsilon}{2K_f} + \frac{\epsilon}{2M_g} \cdot M_g \\
&= \epsilon
\end{aligned}$$