DISSERTATION

GENERALIZED BOOK EMBEDDINGS

Submitted by

Shannon Brod Overbay

Department of Mathematics

In partial fulfillment of the requirements

for the degree of Doctor of Philosophy

Colorado State University

Fort Collins, Colorado

Summer 1998

COLORADO STATE UNIVERSITY

May 13, 1998

WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY SHANNON BROD OVERBAY ENTITLED GENERALIZED BOOK EMBEDDINGS BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

Committee on Graduate Work

Adviser

Department Head

ABSTRACT OF DISSERTATION

GENERALIZED BOOK EMBEDDINGS

An n-book is formed by joining n distinct half-planes, called pages, together at a line in 3-space, called the spine. The book thickness $bt(G)$ of a graph G is the smallest number of pages needed to embed G in a book so that the vertices lie on the spine and each edge lies on a single page in such a way that no two edges cross each other or the spine. In the first chapter, we provide background material on book embeddings of graphs and preview our results on several related problems.

In the second chapter, we use a theorem of Bernhart and Kainen and a result of Whitney to present a large class of two-page embeddable planar graphs. In particular, we prove that a graph G that can be drawn in the plane so that G has no triangles other than faces can be embedded in a two-page book.

The discussion of planar graphs continues in the third chapter where we define a book with a tree-spine. Specifically, we examine the problem of embedding a planar graph in a one-page tree book with a tree-spine having the least number of endvertices. This minimum number of endvertices is called the leaf number of the graph. We construct graphs showing that leaf numbers can be made arbitrarily large for planar graphs. We also give a theorem that provides a bound on the leaf number of a planar graph with a given number of separating triangles.

The fourth chapter focuses on generalized books with modified pages. In this chapter we define the cylinder book, the torus book, and two types of Möbius books. We give general properties of these books and give optimal embeddings of several graphs in these books.

In the fifth chapter, we examine standard book embeddings and generalized book embeddings of Cartesian products of graphs. We expand upon the work of Bernhart and Kainen involving book embeddings of the Cartesian product of an arbitrary graph with a bipartite graph.

Shannon Brod Overbay
Department of Mathematics
Colorado State University
Fort Collins, Colorado 80523
Summer 1998

ACKNOWLEDGEMENTS

I would like to thank my adviser, Dr. Bennet Manvel, for his infinite patience and guidance over the past six years. His encouragement and understanding has made the completion of this dissertation possible.

I would also like to thank my parents for their support of my academic career. As a woman in mathematics, my mother has been a positive role model for me throughout my education. My father, who taught me to follow my dreams, has also been an inspiration to me in my decision to become a teacher.

Special thanks go out to my children, Collin and Maiah. They have provided me with many smiles and hugs that make the work worthwhile.

I am especially grateful to my husband, Darrell, for his invaluable assistance with the difficult graphics and technical details of this paper. I thank him for his love and support throughout graduate school and for the many sacrifices he has made on my behalf.

TABLE OF CONTENTS

PRELIMINARIES

The following definitions have been adapted from Van Lint and Wilson [30] and from Aigner [1]. Other terminology will be defined as needed within the text of this dissertation.

An undirected **graph** G consists of a finite set $V(G)$ of **vertices**, a finite set $E(G)$ of **edges**, and a mapping associating to each edge e an unordered pair $\{u, v\}$ of vertices. If $u = v$, then we call edge e a **loop**. If more than one edge is associated with the vertex pair $\{u, v\}$, then the graph G is said to have **multiple edges**. Graphs without loops and multiple edges are called **simple**. We say that e **joins** its **endpoints** u and v. If v is an endpoint of e, then v and e are called **incident**. Vertices of G are called **adjacent** if they are joined by an edge. The **degree** of a vertex v, denoted $d(v)$, is the number of edges incident with v (loops count twice).

A sequence $v_0, v_1, v_2, \ldots, v_t$ of distinct vertices (except possibly $v_0 = v_t$) with $\{v_{i-1}, v_i\} \in E(G)$ for $i = 1, 2, \ldots, t$, is called a **path** of **length** t in G. The **distance** $d(u, v)$ between two vertices u and v in a connected graph is the length of the shortest path from u to v. A path with $v_0 = v_t$ is called a **circuit** of length t or a **t-cycle**. A **Hamiltonian circuit** is a circuit that passes through every vertex of the graph.

A graph is **connected** if there is a path between any two vertices. Otherwise the graph is **disconnected**, consisting of more than one **connected components**. A connected graph without circuits is a **tree**. A vertex of degree one in a tree is called a **leaf**. A **subgraph** H of a graph G is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We call H a **spanning subgraph** when $V(H) = V(G)$. A spanning subgraph of a connected graph G that is also a tree is a **spanning tree** of G.

Subgraphs may be formed by removing edges and vertices. The graph $G - e$ formed by **removing** an edge e from a graph G is the subgraph with vertex set $V(G)$ and edge set $E(G) - e$. When we remove a vertex v from a graph, we remove both v and all edges incident with v to form the subgraph $G - v$. The graph $G - A$ is the subgraph of G formed by the removal of the vertex set $V(A) \subseteq V(G)$. The **connectivity number** $\kappa(G)$ of a graph G is the smallest number of vertices whose removal disconnects the remaining graph. If G is a graph with $\kappa(G) = 1$ and v is a vertex of G whose removal disconnects G, then v is called a **cut-vertex** of G. Connected graphs have $\kappa(G) \geq 1$ and graphs with $\kappa(G) \geq 2$ are called **biconnected**. In general if $\kappa(G) \geq n$, we say that G is n**-connected**. If $T = \{v_0, v_1, v_2\}$ is a 3-cycle in G so that $G - T$ is disconnected, then T is called a **separating triangle** in G.

We can combine existing graphs to form new graphs. The **union** of the graphs G_1 and G_2 is the graph $G = G_1 \cup G_2$ with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. The **Cartesian product** $G_1 \times G_2$ of G_1 and G_2 is the graph G with vertex set $V(G) = \{(v_1, v_2) |\ v_1 \in V(G_1) \text{ and } v_2 \in V(G_2)\}$. Two vertices (v_1, v_2) and (u_1, u_2) are adjacent in $G_1 \times G_2$ if $u_1 = v_1$ and $\{u_2, v_2\} \in E(G_2)$, or if $u_2 = v_2$ and $\{u_1, v_1\} \in E(G_1)$.

A graph may be represented by a drawing in which each vertex corresponds to a point in the plane and each edge to a line segment or arc connecting the endpoints. If there exists a drawing of G so that no two edges cross, G is called **planar**. Such a drawing is a **planar representation** of G. When drawn in the plane, a planar graph divides the plane into **regions** or **faces**, one of which is unbounded. If every face of a planar representation of G is a triangle, G is called a **maximal** planar graph.

A graph can be drawn in the plane without edge crossings if and only if it can be drawn on the surface of the 3-dimensional sphere without edge crossings. A sphere S_0 is said to have **genus** 0. We form the genus-h surface S_h by adding h handles or holes to the sphere. For example, S_1 is the torus. The **genus** $\gamma(G)$ of G is the smallest h so that G can be drawn on S_h without edge crossings. Such a drawing is called an **embedding** of G.

A simple graph with n vertices and all possible $\binom{n}{2}$ edges is called the **complete graph** K_n. A graph G is **bipartite** if $G = \emptyset$ or if there is a partition $V(G) = V_1 \cup V_2$ of the vertices of G so that every edge of G has one endpoint in V_1 and the other endpoint in V_2. Equivalently, G is bipartite if and only if every circuit of G has even length. The **complete bipartite graph** $K_{m,n}$ is the bipartite graph with $n + m$ vertices $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ and all mn edges $\{a_i, b_j\}$.

The graph consisting only of the vertices and edges of an n-cycle is called a **cycle graph** C_n. The tree with exactly one vertex of degree n and n vertices of degree one is an n-**star**. The n-**dimensional cube graph** Q_n has as vertices all n-tuples $(a_1, a_2, \ldots, a_n)$ with $a_i \in \{0, 1\}$. Two vertices $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ of Q_n are joined by an edge if they differ in exactly one coordinate.

A **proper vertex coloring** of a graph G is a mapping that assigns a color to each vertex of G so that no two adjacent vertices have the same color. The smallest number of colors needed for a proper vertex coloring of G is the **chromatic number** $\chi(G)$ of G. Similarly, a **proper edge coloring** of G is an assignment of colors to edges so that no two edges with a common endpoint have the same color. The minimum number of colors needed for such an edge coloring is called the **chromatic index** $\chi'(G)$ of G.

Chapter 1

INTRODUCTION

The book embedding problem was first introduced by Kainen in 1973 (see [16]). In an article published in 1979, Kainen and Bernhart [2] laid the groundwork for further study of book embeddings of graphs. They defined an n**-book** as a line L in 3-space, called the **spine**, and n half-planes, called **pages**, with L as their common boundary. An n**-book embedding** of a graph G is an embedding of G in an n-book with the vertices of G on the spine and each edge of G within a single page so that no two edges cross. The **book thickness** $bt(G)$ or **page number** $pg(G)$ of a graph G is the smallest n so that G has an n-book embedding.

In their 1979 paper, Bernhart and Kainen raised several questions about book embeddings. First, what are some basic properties of book embeddings? Bernhart and Kainen compare book thickness with other graph invariants and give bounds for the book thickness of graphs with n vertices and q edges. They also show the equivalence of the book embedding problem and a circular embedding problem. Second, what is the book thickness of a given graph? They answer this question for K_n and give bounds for the book thickness of $K_{m,n}$ and Q_n. They also examine bounds for the book thickness of Cartesian products of some graphs and they give characterizations of one and two-page embeddable graphs. Finally, what is the relationship between genus and book thickness? In an attempt to answer this question, Bernhart and Kainen present graphs with $bt(G) = 3$ but arbitrarily large genus. In the other direction, they conjecture that there are graphs with fixed genus that have arbitrarily large book thickness. In particular, they conjecture that there are planar graphs with arbitrarily large book thickness.

Several others have worked to settle the question of genus versus book thickness. In 1984 Bernhart and Kainen's conjecture for planar graphs was refuted. Using a result of Whitney [31], Buss and Shor [4] showed that all planar graphs can be embedded in nine pages. With a different construction, Heath [11] reduced this number to seven pages. Extending the techniques of Heath, Yannakakis finally settled the book thickness problem for planar graphs in 1986 by showing that four pages are necessary and sufficient (see [32] and [33]).

The question of whether book thickness can be arbitrarily large for graphs with fixed genus remained open for graphs with genus $g > 0$ until 1987 when Heath and Istrail [12] disproved the conjecture by giving an algorithm that embeds any genus g graph in $O(g)$ pages. In their paper, Heath and Istrail suggest that this result could be improved to $O(\sqrt{g})$ for genus g graphs. In 1988, Malitz [19] built on the work of Heath and Istrail to give an $O(\sqrt{g})$-page embedding for any genus g graph.

The book embedding problem has several applications (see [3], [4], [5], [6], [7], [17], [20], [21], [23], [25], [26], and [27]). Chung, Leighton, and Rosenberg developed the Diogenes method of designing fault-tolerant VLSI processor arrays (see [5], [6], [25], and [26]). This method involves laying out processing elements (the vertices) in a line (the spine). Non-faulty processing elements are connected by wires (edges) and bundles of non-crossing wires correspond to pages. Other applications mentioned by Chung, Leighton, and Rosenberg include sorting with parallel stacks, single-row routing, and applications to complexity theory.

The book thickness of particular graphs has also been the focus of many articles (see [2], [4], [5], [6], [7], [22], [23], and [24]). In general this is a difficult question. Garey, Johnson, Miller, and Papadimitriou [8] show that the problem of determining whether an arbitrary graph is k-page embeddable is NP-complete, even with a pre-specified vertex ordering. However, the exact book thickness and good bounds for book thickness are known for several classes of graphs. Motivated by the applications of book embeddings, Chung, Leighton, and Rosenberg give optimal 2-page embeddings of square grids and X-trees (see [5] and [6]). They also include bounds on Q_n and the Benes network graph $B(n)$. Games [7] improves the results for $B(n)$, and gives optimal 3-page book embeddings of this graph as well as for the FFT (Fast Fourier Transform) network and the barrel shifter network. Muder, Weaver, and West [22] improve Bernhart and Kainen's bounds for the book thickness of $K_{m,n}$. More recently, Obrenic [23] has given algorithms to embed de Bruijn and shuffle-exchange graphs in five pages, providing the first nontrivial bounds for the book thickness of these graphs.

Some extensions and applications of book embeddings involve the parameter of **pagewidth**. In a book embedding of a graph, the **width** of a page is the maximum number of edges that cross any line perpendicular to the spine. The **pagewidth** of a book embedding is the maximum width of any page. Many authors examine trade-offs between book thickness and pagewidth. Chung, Leighton, and Rosenberg [6] present a class of n-vertex graphs in which every one-page embedding requires one page of width $n/2$, but for which there exist two-page embeddings with pagewidth two. This tradeoff has also been examined by Heath [10] and Stöhr [27]. Heath provides an algorithm for embedding one-page graphs with maximum degree d in two-page books with pagewidth $O(d \log n)$. Stöhr describes for $n \geq 3$ a family of graphs having n-page embeddings, but unbounded page width. However, with the addition of one more page, the page width is bound by a constant.

One generalization of the book embedding problem is the **black/white book embedding problem**. In this problem, a graph G and a partition of the vertex set V of G into a set U of black vertices and a set $V - U$ of white vertices is given. A black/white book embedding of G is a book embedding of G with the constraint that the vertices of U are placed consecutively on the spine. This problem, studied by Moran and Wolfstahl, arises in VLSI design applications where there needs to be a separation between input ports (black vertices) and output ports (white vertices) of a VLSI chip (see [20] and [21]).

Others have considered the book embedding problem with more complex restrictions on vertex-ordering, such as the poset embedding problem. If L is a linear extension of a partially ordered set (poset) $P = (P, \leq)$ (i.e. L is a total order containing P), then a book embedding of P with respect to L is a book embedding of the Hasse diagram $H(P)$ of P so that the elements of P are placed on the spine in accordance with L. The **page number** of P is defined to be the least number of pages required to embed $H(P)$ in a book where the vertex ordering is taken over all linear extensions of P. In general, the book thickness of planar posets is unknown. Hung [14] discusses the poset embedding problem and constructs a planar poset which requires four pages. In his paper, Hung also gives an upper bound of six pages for a class of planar posets previously thought to have unbounded book thickness.

In the second chapter of this dissertation, we investigate graphs with book thickness $bt(G) \leq 2$. The problem of determining whether an arbitrary graph is two-page embeddable is *NP*-complete (see [6] and [8]). However, the characterizations of one and two-page embeddable graphs given by Bernhart and Kainen [2] allow the classification of many planar graphs. We present several graphs known to have book thickness $bt(G) \leq 2$ and discuss the relationship between the two-page embedding problem and the Hamiltonian circuit problem. Using a result of Whitney [30], we are able to demonstrate a large class of two-page embeddable graphs.

In the standard book embedding problem, the spine is a straight line in 3-space. In the third chapter, we consider generalized book embeddings of planar graphs in books with tree-shaped spines. The vertices of the graph are placed along the tree-spine and the edges are drawn in the plane without crossing each other or the spine. Since every connected planar graph has a spanning tree, it is easy to see that for any planar graph G there is a tree on which G can be embedded. We examine the problem of embedding a graph on a tree with the smallest number of endvertices. In this discussion, we construct graphs to show that this number can be made arbitrarily large.

In the fourth chapter, we examine generalized books. First, we consider a book where the spine is a line in 3-space, but the pages are modified. We allow the pages to wrap around and reconnect at the spine, forming cylindrical pages. Next, the spine is realized as a ring on a torus, allowing wrapping of edges in two directions on the toroidal pages. We present some general properties of these two books and we give optimal embeddings for several graphs. Finally, we explore a variation, presented by Kainen [16], which allows a twist in the spine. We present graphs in which the change in orientation of the Möbius book greatly reduces the number of pages required for an embedding.

In the fifth chapter, we examine topics related to book embeddings of Cartesian products of graphs introduced in [2] and [15]. Using the results of chapter four, we provide methods for embedding the Cartesian product of a graph G and an odd cycle in a torus book. We conclude with a discussion of the issue of dispersability, which arises in an attempt to attain good bounds for the book thickness of Cartesian products of graphs. A graph G with maximum degree k is called **dispersable** if there is a proper k-edge coloring of G and a k-page book embedding of G so that all edges of one color lie on the same page. It is unknown whether all bipartite graphs are dispersable. Although unable to answer this question, we give some insight into the solution of this problem and we present dispersable book embeddings for several classes of bipartite graphs.

Chapter 2

GRAPHS WITH BOOK THICKNESS $BT(G) \leq 2$

Recall that an n**-book** is a set of n half-planes in 3-space which meet along a common line (the **spine**). The **book thickness** $bt(G)$ of a graph G is the minimum number of pages needed to embed the graph G in a book so that the vertices lie on the spine and each edge lies on a single page in such a way that no two edges cross. In this chapter we will examine graphs with $bt(G) \leq 2$.

The only graphs with $bt(G) = 0$ consist entirely of isolated vertices, since each edge of a graph must be assigned to a page. Observing that the vertices of each connected component $C_1, C_2, \ldots, C_k$ of a disconnected graph G can be grouped by component along the spine, it follows that $bt(G) = \mathbf{max}\ \{bt(C_1), bt(C_2), \ldots, bt(C_k)\}$. Hence, from this point forward, all graphs are assumed to be connected. Loops and multiple edges also do not complicate the book embedding problem. In a book embedding, a loop can be placed next to the spine and a single edge can be replaced by multiple copies without causing edge crossings. For simplicity, we will also restrict our discussion to simple graphs.

It is easy to see that the set of one-page embeddable graphs includes paths. We embed the vertices along the spine according to the natural ordering of the path, $v_0, v_1, \ldots, v_n$. Now all edges $\{v_i, v_{i+1}\}$ can be placed on a single page without crossing (see Figure 2.1). Not only paths, but all trees admit one-page embeddings. This is shown by induction on the number of vertices in the tree.

Figure 2.1 *One-page book embedding of the path of length n.*

Theorem 2.1 *If T is a tree, then $bt(T) \leq 1$.*

Proof : Let T be a tree. If $|V(T)| = 1$, place the single vertex on the spine.

Now suppose the theorem holds for all trees with $|V(T)| = 1, 2, \ldots, k-1, k \geq 2$. Consider tree T with $|V(T)| = k$. Since $k \geq 2$, T must have at least one leaf, v. Removing v and its adjoining edge e results in a tree $T - v$ with $k - 1$ vertices. By induction, we may now embed $V - v$ in a book with one or fewer pages. Let u be the unique vertex of T adjacent to v. Then u must lie on the spine in the book embedding of $V - v$. Place v on the spine to the immediate right of u. Since edges of $V - v$ lie only on the pages, this placement will not conflict with the existing book embedding of $V - v$. We may now draw edge e between u and v below any edges on the page, avoiding crossings with other edges adjacent to u. This gives the desired one-page embedding of T (see Figure 2.2). ■

Figure 2.2 *One-page book embedding of the height three binary tree.*

Figure 2.2 shows a one-page embedding of the complete binary tree of height three. By the above theorem, we see that graphs without circuits are one-page embeddable. However, there are clearly graphs with circuits that also have book thickness one. We can embed the circuit of length n in a one-page book just as we embedded the path of length n. In the case of the circuit, the additional edge $\{v_0, v_n\}$ can be placed above the other edges in the page, without crossing, as illustrated by the dotted line segment in Figure 2.1.

Now suppose the vertices of an arbitrary graph are ordered $v_1, v_2, \ldots, v_n$ along the spine of a book. The edges of the circuit $v_1, v_2, \ldots, v_n$ can be added to any page of the book without causing edge crossings in a simple way. We place the edges of the path $v_1, v_2, \ldots, v_n$ close to the spine and the edge $\{v_0, v_n\}$ above the other edges on the page. Every edge on a particular page lies within or is on this outer circuit.

If we stretch this circuit into a circle, the problem of determining whether a given graph can be embedded in a k-page book can be viewed in terms of a circular embedding problem. Embedding a graph G in a k-book is equivalent to placing the vertices in a circle and coloring the edges (represented by chords of the circle) with k colors so that no two edges of the same color cross. With this circular view of the spine, it is also now clear that if G is embeddable in a k-page book with vertex-ordering $v_1, v_2, \ldots, v_n$ along the spine, then any cyclic permutation of the vertices along the spine also gives a k-book embedding of G.

The circular realization of the spine allows us to give an alternate description of graphs with book thickness k. A graph G is called **outerplanar** if it can be drawn in the plane so that all vertices of G lie on the same face. Equivalently, G is outerplanar if all the vertices of G can be placed in a circle in such a way that all edges of G are non-crossing chords of the circle. This leads to the following result (see [2] and [10]).

Theorem 2.2 *A graph G has a k-page embedding with vertex ordering $v_1, v_2, \ldots, v_n$ if and only if $G = G_1 \cup G_2 \cup \ldots \cup G_k$, where each G_i is an outerplanar graph embedded with vertex-ordering $v_1, v_2, \ldots, v_n$.* ■

Now the following characterization of one-page embeddable graphs given by Bernhart and Kainen [2] is clear.

Theorem 2.3 *$bt(G) \leq 1$ if and only if G is outerplanar.* ■

Large classes of outerplanar, and thus one-page embeddable, graphs are known (see Syslo [28]). There are many examples of graphs that are planar but not outerplanar. A simple one-page embeddable graph with n vertices has at most $2n - 3$ edges, since it can have at most n edges for a completed outer n-cycle at most $n - 3$ edges (corresponding to a complete triangulation) in the interior of that n-cycle. The graph K_4 is the smallest example of a graph that is not outerplanar. K_4 has $n = 4$ vertices and 6 edges, which exceeds the upper bound of $2(4) - 3 = 5$ edges. Although it is not one-page embeddable, K_4 does admit a two-page embedding (see Figure 2.3).

Figure 2.3 *Two-page book embedding of K_4.*

A two-page book consists of two half-planes that meet at the spine. This may be realized by drawing a straight line L in the plane for the spine. The two pages correspond to the half-planes above and below L. Thus it is clear that any two-page embeddable graph is planar. Is the converse true? Does every planar graph have a two-page embedding? Bernhart and Kainen [2] give a characterization of two-page embeddable graphs which helps answer this question.

Theorem 2.4 *$bt(G) \leq 2$ if and only if G is a subgraph of a planar Hamiltonian graph.*

Proof: Let G be a graph with $bt(G) \leq 2$. Consider a two-page book embedding of G. The desired Hamiltonian circuit is found by following the natural ordering of the vertices along the spine, adding any missing edges to form the outer circuit. With the added edges, we now have a planar Hamiltonian graph.

Conversely, suppose G is a subgraph of a planar Hamiltonian graph G'. Draw G' in the plane and trace out a Hamiltonian circuit C in G'. The circuit C together with the edges inside C form one page and the edges outside C form the second page. Now we have a two-page embedding of G' which induces the desired two-page book embedding of G. ∎

Graphs that are subgraphs of planar Hamiltonian graphs are called **subhamiltonian**. Planar Hamiltonian graphs are clearly subhamiltonian, and thus two-page embeddable. We have large classes of two-page embeddable graphs due to the following results of Whitney and Tutte.

Theorem 2.5 *A maximal planar graph without separating triangles has a Hamiltonian circuit.*

Proof: See Whitney [31]. ■

Theorem 2.6 *A* 4*-connected planar graph with at least two edges has a Hamiltonian circuit.*

Proof: See Tutte [29]. ■

Since there exist maximal planar graphs that are not Hamiltonian, there are planar graphs that are not two-page embeddable. Maximal planar graphs without separating triangles are embeddable in two-page books. Thus, separating triangles are critical in forming maximal planar graphs that need more than two pages. The **stellation** $St(G)$ of a planar graph G is formed as the result of placing a new vertex in every face (including the outer face) of G and connecting it to each vertex around the face. We can repeat this process by letting $St^n(G) = St(St^{n-1}(G))$. The maximal non-Hamiltonian planar graph $St^2(K_3)$ is shown in Figure 2.4.

Figure 2.4 *The second stellation of the triangle $St^2(K_3)$.*

In their 1979 paper, Bernhart and Kainen conjecture that the book thickness of graphs with fixed genus, particularly of planar graphs, is unbounded. Specifically, they suggest that $bt(St^n(G))$ can be made arbitrarily large if G is any maximal planar graph. Heath [11] disproves the specific claim by showing that $St^n(K_3)$ are all embeddable on three pages. Figure 2.5 depicts a 3-page book embedding of $St^2(K_3)$. Several authors have disproved the larger conjecture for planar graphs (see [4], [11], [12], [19], [32], and [33]) by giving various finite bounds for the book thickness of a planar graph. Yannakakis settles the issue for planar graphs by offering a best bound of four pages (see [32] and [33]).

Figure 2.5 *Three-page book embedding of $St^2(K_3)$.*

In his argument, Yannakakis uses techniques of Heath [11] to break a planar graph into levels. Yannakakis begins with a planar graph in which every vertex lies on or within an outer circuit C and all faces inside C are triangles. Vertices on the outer face are said to be at level 0, those at distance one from any vertex on the outer face are at level 1, and so on. To obtain the vertex-ordering on the spine, the vertices at level 0 are placed on the spine in a clockwise ordering corresponding to the ordering of C. The level 0 vertices are deleted so that the remaining graph consists of level 1 nodes on the outer face. Next, level 1 vertices are placed among the level 0 vertices on the spine. The circuits comprised of level 1 vertices are lined up on the spine with a counter-clockwise ordering. The vertices interior to each of the level 1 circuits are now placed on the spine in a recursive fashion. The ordering of the vertices (clockwise or counter-clockwise) is reversed for circuits at each level. After all vertices are positioned on the spine, edges are carefully assigned to four pages, giving the following result.

Theorem 2.7 *If G is a planar graph, then $bt(G) \leq 4$.*

Proof: See Yannakakis [32]. ∎

In his paper, Yannakakis also outlines the construction of a planar graph which needs four pages. Hence, four pages are also necessary to accommodate all planar graphs. Since the example of Yannakakis is extremely large and complex and it is the only published example needing four pages, it appears that three pages are sufficient for most small planar graphs. If the original triangulation has sparse separating triangles, Kainen [16] suggests that only three pages are needed for the book embedding. By Whitney's theorem and Theorem 2.4, we also know that if a graph is a subgraph of a maximal planar graph without separating triangles, then it is embeddable in a 2-book. Using this observation, we present large classes of two-page embeddable graphs in the next two theorems.

First, we show that a 3-connected planar graph without separating triangles can be embedded in a two-page book. We then extend this result to show that any planar graph G that can be drawn in the plane in such a way that the only triangles of G are faces can be embedded in a two-page book. In the following proofs, we add a sequence of vertices and edges to the original graph G to produce a maximal planar graph without separating triangles that contains G.

Theorem 2.8 *A 3-connected planar graph without separating triangles is subhamiltonian.*

Proof: Let G be a 3-connected planar graph without separating triangles drawn in the plane. We will form a new graph G^* by placing a vertex in the interior of each non-triangular face and connecting it to each vertex of the circuit forming the face (i.e. we stellate each non-triangular face of G) as shown in Figure 2.6.

Figure 2.6 *Stellation process on the Herschel graph.*

It is clear that G^* is a planar graph, since the new edges appear only inside the faces of G and do not conflict with existing edges. The graph G^* is also maximal since each non-triangular face of G bounded by an n-cycle is transformed into n triangular faces. Now we claim that G^* has no separating triangles.

Suppose, by way of contradiction, that G^* contains a separating triangle T. Since G had no separating triangles, at least one vertex v of T must have been added in the formation of G^*. But, each vertex v added to G is adjacent only to the vertices of the n-face inside which v was placed. Thus, the other two vertices x and y of T were in the original graph. Since T is a separating triangle in G^*, the vertex set $\{x, y\}$ forms a separating two-set in $G^* - v$.

Noticing that the removal of v just gives us the original n-face back, we see that G is a subgraph of $G^* - v$. We now show that the vertex set $\{x, y\}$ must also be a separating two-set in G by observing that each connected component of $(G^* - v) - \{x, y\}$ must contain at least one vertex of the original graph G. Suppose not. Then there is a connected component of $(G^* - v) - \{x, y\}$ consisting only of newly added vertices and edges. Since no two added vertices are adjacent, the only possibility is that some connected component is a single added vertex. However, this cannot happen because each added vertex is adjacent to at least four vertices of G. Hence, the removal of only two vertices of G cannot isolate any added vertex. Now we see that the number of connected components of $G - \{x, y\}$ must be at least as great as the number of connected components of $(G^* - v) - \{x, y\}$. Hence, the vertex set $\{x, y\}$ was a separating two-set of G. Now we have a contradiction of the assumption that G was 3-connected.

It follows that G^* is a maximal planar graph without separating triangles. We now appeal to Theorem 2.4 and Whitney's theorem (Theorem 2.5) to give a two-page book embedding of G^*. Now all added vertices can be removed, resulting in the desired book embedding of G in a two-page book. ■

Theorem 2.8 is of interest since there exist non-Hamiltonian planar graphs that are 3-connected without separating triangles. The graph in Figure 2.6 is called the Herschel graph (see [1], p.130). It is the smallest non-Hamiltonian 3-connected planar graph. The bold-faced edges in Figure 2.7 represent a Hamiltonian ordering of the vertices along the spine in a two-page embedding of the Herschel graph. The methods of Theorem 2.8 provide two-page book embeddings of other such graphs, despite their possible lack of a Hamiltonian circuit.

Figure 2.7 *A Hamiltonian circuit in the stellation of the Herschel graph.*

We now extend the results of Theorem 2.8 to any planar graph G that has a planar drawing so that triangles of G appear only as faces. These graphs are not necessarily 3-connected. First, we prove the following lemma which allows us to transform a biconnected planar graph having no triangles other than faces into a 3-connected planar graph without separating triangles. We apply this process to each biconnected component of the graph and use Theorem 2.8 to give two-page book embeddings each of these components. Finally, in Theorem 2.9, we show how to put the two-page embeddings of the biconnected components together to give a two-page book embedding of the original graph.

Lemma 2.1 *Let G be a biconnected planar graph that can be drawn in the plane so that the only triangles of G are faces of G. Then G is a subgraph of a 3-connected planar graph without separating triangles.*

Proof: Let G be a biconnected planar graph that can be drawn in the plane so that the only triangles of G are faces of G. We will show that G is a subgraph of a 3-connected planar graph by induction on the number of separating sets of two vertices (or separating two-sets) in G. If G has no separating two-sets, then G is 3-connected, so there is nothing to show.

Now suppose the theorem holds for all planar graphs with $s = 0, 1, 2, \ldots, k-1$ separating two-sets. Let G be a biconnected planar graph with k separating two-sets drawn in the plane so that there are no triangles of G other than faces. Let $A = \{u, v\}$ be a separating two-set of vertices of G. We will add a sequence of vertices and edges to G to produce a new graph G' having fever separating two-sets. Specifically, A will no longer be a separating two-set in G'. The result will follow by induction.

Let $G_1, G_2, \ldots, G_n$ be the connected components of G which remain upon the removal of A. Assume that the G_is are labeled according to the natural clockwise ordering about v in the planar embedding of G. We see that this ordering around v exists by the planarity of G and since each G_i must be connected to both v and u. It is possible that there is an edge connecting u and v in G. If so, without loss of generality, assume it lies between G_n and G_1. We will now form a chain connecting each G_i to G_{i+1} for $i = 1, 2, \ldots, n-1$. Then $G - A$ will consist of only one component.

For $i = 1, 2, \ldots, n-1$, we connect G_i to G_{i+1} to form the new graph G' in the following way. Let x_i be the last vertex of G_i that is adjacent to v and let w_i be the first vertex of G_{i+1} that is adjacent to v, with respect to the clockwise ordering around v. We add a new vertex v_i to the graph between x_i and w_i in the plane. We now add three edges to the graph by joining v_i to v, x_i, and w_i. By our choice of x_i and w_i, there is no edge adjacent to v which lies between x_i and w_i in the planar embedding prior to the addition of v_i. Thus, we can insert v_i and the three associated edges without violating planarity (see Figure 2.8).

The addition of v_i with the three edges does not add any triangles to the graph, other than the two triangular faces $T = \{v_i, x_i, v\}$ and $S = \{v_i, w_i, v\}$. The only other possible triangle with v_i is the triangle with the vertices v_i, x_i, and w_i. But, there cannot be an edge between x_i and w_i in the graph, otherwise x_i and w_i would not be in different components of $G - A$. Hence, the graph G' does not have any triangles other than faces. We now need to show that G' is biconnected, with fewer separating two-sets than G.

First, G' is clearly connected, since G was connected and since each added vertex is adjacent to vertices of G. Second, G' is biconnected. The removal of a single vertex x of $V(G)$ cannot cause a separation of G'. For, if two vertices of G were in separate components of $G' - x$, then they would be in separate components of $G - x$, contradicting the fact that G was biconnected. The only other possibility is that a component of $G - x$ consists of a single vertex of $V(G' - G)$. But each added vertex is adjacent to three distinct vertices of G. Hence, the removal of one vertex of $V(G)$ cannot isolate a newly added vertex. Also, the removal of any vertex v_i of $V(G' - G)$ cannot separate G'. This can be seen since v_i is connected to only the three vertices x_i, w_i, and v in the connected graph G'. Since $P = x_i, v, w_i$ is a path in both G and $G' - v_i$, these three vertices all lie in the same component of $G' - v_i$. Hence, the removal of an added vertex v_i cannot separate G'.

Figure 2.8

Now we have that G' is a biconnected graph drawn in the plane so that there are no triangles other than faces. All that remains is to show that G' has fewer separating two-sets than G. To do this, we will first show that fewer than k of the original separating two-sets of G are also separating two-sets of G'. Then, we will show that none of the vertices v_i added to G in the construction of G' can be part of a separating two-set of G'.

By our construction of G', $A = \{u, v\}$ is not a separating two-set of G'. This follows since the graph $G' - A$ consists of a single component formed by the components $G_1, G_2, \ldots, G_n$ of G linked together by the paths $P_i = x_i, v_i, w_i$ for $i = 1, 2, \ldots, n-1$. Next, we claim that any other separating two-set of G' consisting of two vertices $B = \{y, z\}$ of $V(G)$ must be a separating two-set of G. Consider the graph $G' - B$. No component of $G' - B$ contains a single added vertex v_i since each v_i touches three vertices of G. And since the added vertices are only connected to vertices of G, it follows that each connected component of $G' - B$ contains vertices of G. Hence, B is a separating two-set of G.

We now show that no added vertex v_i is a member of a separating two-set of G'. Clearly, no two added vertices v_i and v_j can form a separating two-set in G' since their removal leaves the connected graph formed by G and the $n-3$ added vertices, each connected to three vertices of G. The remaining possibility is a separating two-set consisting of one added vertex v_i and a vertex x of $V(G)$. As shown above, the removal of v_i leaves a connected graph $G' - v_i$. If two vertices of G are in separate components of the graph formed by removing x from $G' - v_i$, then x was a cut-vertex of G, contradicting the biconnectedness of G. Otherwise, one component of the graph $G' - \{x, v_i\}$ must consist of a single added vertex. Again, this cannot happen since each added vertex is adjacent to three vertices of G. Thus, no added vertex contributes to a separating two-set of G'.

Now we see that G' has fewer separating two-sets than G. Hence, G' is a biconnected planar graph, drawn in the plane with no triangles other than faces, that has less than k separating two-sets. By induction, G' is a subgraph of a 3-connected planar graph without separating triangles. Since G is a subgraph of G', the same is true for G. ■

With Lemma 2.1 we are able to take a biconnected planar graph without any triangles other than faces and use the results of Theorem 2.8 to give two-page book embeddings for each of these components. Finally, we string the embeddings of the biconnected components together to give a two-page book embedding of the original graph.

Theorem 2.9 *A planar graph G that can be drawn in the plane so that the only triangles of G are faces of G is subhamiltonian.*

Proof: Let G be a graph satisfying the conditions of the theorem. First, we can assume that G is connected. Otherwise we can apply the methods of this theorem to each connected component. Second, we show that if the result holds for the biconnected components of G, then it holds for G.

To see this, consider a cut-vertex v of G. Let $G_1, G_2, \ldots, G_n$ be the connected components of $G - v$. We now form the graphs G'_i by adding v, along with the edges of G between v and G_i, to each component G_i as shown in Figure 2.9. Since there are no edges between any of the G'_is, we embed each of these components separately.

If each G'_i is biconnected, we build a book embedding of G from book embeddings for each G'_i as follows. First, we recall a previous observation that any cyclic permutation of the vertices along the spine also yields a valid book embedding.

Figure 2.9

Hence, we can cycle the vertices of each G'_i so that v is the left-most vertex. We now place v on the spine. To the right of v, we order the components $G'_1, G'_2, \ldots, G'_n$ from left to right along the spine. Next, we remove the extra copies of v from each of the G'_is, converting them back to the original components G_i of $G - v$. Now we will reconnect the single vertex v to each G_i to give the desired book embedding of G. This can be done by placing edges between G_{i+1} and v above the edges between G_i and v (see Figure 2.10). Hence, all edges between v and the components of $G - v$ fit on a single page without crossing.

If any G'_i has a cut-vertex, then we perform the same procedure on G'_i. This process must eventually terminate since G only has a finite number of cut-vertices. In this process, all edges between a cut-vertex v and the components connected to v all fit on one page. It is now clear that a graph G admits a k-page embedding ($k \geq 1$) if and only if each of the biconnected components of G are k-page embeddable. It follows that G is subhamiltonian if and only if the biconnected components of G are subhamiltonian.

Figure 2.10

By Lemma 2.1, we know that each biconnected component of G can be augmented, by adding a sequence of vertices and edges, to a 3-connected planar graph without separating triangles. Then, by Theorem 2.8, these 3-connected graphs are subhamiltonian. Hence, each biconnected component of G is subhamiltonian. Thus, G is subhamiltonian. ∎

In showing that biconnected planar graphs without non-face triangles are subhamiltonian we add both edges and vertices to the original graph. In the following corollary we show that two-page embeddings of such graphs can be reached by adding edges only.

Corollary 2.1 *A planar graph G that can be drawn in the plane so that the only triangles of G are faces of G can be edge-augmented to a planar Hamiltonian graph.*

Proof: Let G be a planar graph drawn in the plane so that the only triangles of G are faces. We follow the proof of Theorem 2.9 to obtain a two-page book embedding of G, deleting all extra vertices and edges added in the process. We then add any missing edges along the spine to complete the outer Hamiltonian circuit. The resulting graph is a planar Hamiltonian graph formed by adding only edges to the original graph. ■

Corollary 2.1 implies that for any planar graph G drawn in the plane so that the only triangles are faces, there is a sequence of edges that can be added to G to produce a planar Hamiltonian graph. However, the graph formed by this sequence of added edges does not necessarily satisfy the conditions of Whitney's Theorem. Figure 2.11 shows such a graph. It is easy to see that the addition of any edge results in a separating triangle.

Figure 2.11

We have defined a subhamiltonian graph as a subgraph of a planar Hamiltonian graph. Other authors give an alternative definition of a subhamiltonian graph as a spanning subgraph of a planar Hamiltonian graph (see [25], p. 218). Using an argument similar to that of Corollary 2.1, we show that the two definitions are equivalent.

Theorem 2.10 *A graph G is subhamiltonian if and only if it is a spanning subgraph of a planar Hamiltonian graph.*

Proof: Let G be a subhamiltonian graph. Then G admits a two-page book embedding by Theorem 2.4. To this two-page embedding of G we add edges along the spine to complete the outer Hamiltonian circuit. Now we have a planar Hamiltonian graph that has G as a spanning subgraph.

The converse is clear. ■

Another interesting consequence of Theorem 2.9 is the two-page embeddability of bipartite planar graphs.

Corollary 2.2 *If G is a planar bipartite graph, then G is subhamiltonian.*

Proof: Let G be a planar bipartite graph drawn in the plane. Since G is bipartite then G has no odd cycles. Hence, G contains no triangles and satisfies the conditions of Theorem 2.9. The result follows. ■

We observe that the two-page embeddings guaranteed by Corollary 2.2 and Theorem 2.9 are best-possible results. This follows since there are planar bipartite graphs and other graphs satisfying the conditions of Theorem 2.9 that are not outerplanar. For example, the 3-dimensional hypercube Q_3 is a bipartite planar graph. Conditions given in [28] show that Q_3 is not outerplanar. Hence, Q_3 is not embeddable in a one-page book. Figure 2.12 shows a two-page book embedding of Q_3.

To the best of our knowledge, the results of Theorem 2.9 including the two-page classification of planar bipartite graphs, were previously unknown. The set of two-page embeddable graphs resulting from Theorem 2.9 include families of graphs that were previously classified individually. Chung, Leighton, and Rosenberg [5]

Figure 2.12 *Two-page book embedding of Q_3.*

show that square grids and X-trees are two-page embeddable. The two-page embeddability of both families of graphs follow from Theorem 2.9 and Corollary 2.2.

The $n \times n$ **square grid** is the planar graph formed by taking the Cartesian product of two paths of length n (see Figure 2.13). This graph contains no odd cycles. Thus, the square grid satisfies the conditions of Corollary 2.2 and is two-page embeddable.

Figure 2.13 *The 4×4 square grid.*

The **depth-d X-tree** $X(d)$ is the complete binary tree of height d with additional edges going across each level of the tree (see Figure 2.14). An X-tree G can be drawn in the plane so that all triangles of G are faces. Hence, by Theorem 2.9, G has a two-page book embedding.

Figure 2.14 *The depth-4 X-tree $X(4)$.*

Theorem 2.9 gives some insight into the problem of finding Hamiltonian circuits in graphs. The Herschel graph in Figure 2.6 is the smallest example of a non-Hamiltonian planar 3-connected graph. Although not Hamiltonian, it is bipartite and thus subhamiltonian. The bold-faced edges in Figure 2.7 trace the desired Hamiltonian circuit. Even a 3-connected planar graph in which every vertex has degree three (cubic) is not necessarily Hamiltonian. The Tutte graph (see [1], p. 50) is an example of a non-Hamiltonian cubic 3-connected planar graph. However, Tutte's famous graph contains no triangles. Hence, it is subhamiltonian. It is unknown whether all cubic 3-connected planar bipartite graphs are Hamiltonian. Several people have worked to settle this problem more commonly known as Barnette's conjecture (see [13]). Although unable to settle the Barnette conjecture, our results show that such graphs are at least subhamiltonian.

Theorem 2.9 also gives some evidence in support of a conjecture of Chartrand, Geller, and Hedetniemi (see [18]). They suggest that every planar graph can be edge-partitioned into two outerplanar subgraphs. The claim clearly holds for a subhamiltonian graph G. The two pages in a two-page embedding of G form the partition of the edges of G. By proving the two-page embeddability of a large class of planar graphs, we have shown that many planar graphs can be edge partitioned into two outerplanar graphs in a simple way.

An interesting consequence of Corollary 2.2 is that every planar graph is closely related to a two-page embeddable graph in the following way. We say that we **subdivide** an edge $e = \{u, v\}$ if we replace e with a path $u = v_0, v_1, \ldots, v_n = v$. A graph G' is called a **subdivision** of a graph G if G' is formed by subdividing edges of G. Two graphs are called **homeomorphic** if they both be derived from the same graph by performing edge subdivisions. Bernhart and Kainen [2] show that any graph is homeomorphic to a graph that is embeddable in a three-page book. We provide a different argument to show that in the case of planar graphs, this result can be improved to two pages.

Corollary 2.3 *Every planar graph is homeomorphic to a two-page embeddable graph.*

Proof: Let G be a planar graph. We form a planar bipartite graph G^* that is homeomorphic with G by subdividing every edge of G (i.e. we insert a vertex of degree two into each edge of G). Now every circuit of length n in G has length $2n$ in G^*. Since every circuit of G^* is even, it follows that G^* is bipartite and planar. Now Corollary 2.2 provides the desired two-page book embedding of G^*. ■

Corollary 2.3 illustrates an interesting distinction between the problem of embedding graphs on surfaces of genus g and the book embedding problem. In the surface embedding problem, the act of subdividing edges does not affect the genus of a graph. However, subdividing edges reduces the book thickness to at most three for general graphs and at most two for planar graphs.

In this chapter, we have developed large classes of two-page embeddable planar graphs. We showed that if we can add edges and vertices to a graph G to produce a maximal planar graph G' without separating triangles, then G is two-page embeddable. However, there are maximal planar Hamiltonian graphs that have separating triangles. The graph $St^2(K_3)$, shown in Figures 2.4 and 2.5, is the smallest non-Hamiltonian maximal planar graph. But, $St^2(K_3)$ clearly has several separating triangles. In the next chapter, we take a closer look at how separating triangles complicate embeddings of planar graphs.

In Chapter 3, we continue our focus on planar graphs. Planar graphs that are not subhamiltonian, like $St^2(K_3)$, are of particular interest. We know that such graphs can be embedded in books with either three or four pages. However, 3-books and 4-books are non-planar structures. Instead, we embed such graphs on planar structures by modifying the spine. Beyond the standard linear spine, we consider books with spines that are trees.

Chapter 3

EMBEDDING PLANAR GRAPHS IN BOOKS WITH TREE-SPINES

In this chapter, we will continue our study of book embeddings of planar graphs. Instead of the standard book with a line in 3-space as a spine, we form a new type of book by allowing trees, drawn in the plane, as spines. Rather than half-planes, the pages of books with tree-spines are planes.

The graph G is embedded in an **n-tree book** with **tree-spine** T, or an **(n, T)-book**, in the following way. The vertices of G are placed on T and the edges of G are placed in the n plane pages (each containing a copy of T) so that no edge crosses T and no two edges on the same page cross each other. The **book thickness of G with respect to the tree T** $bt(G, T)$ is the smallest n so that G can be embedded in an (n, T)-book.

We focus our attention on $(1, T)$-book embeddings of planar graphs. For a planar graph G, we seek a tree T having the smallest number of endvertices so that G can be embedded in a $(1, T)$-book. We define the **leaf number** $lf(G)$ of a planar graph G to be the smallest number of leaves of all trees T on which G has a $(1, T)$-book embedding. Every connected planar graph G has a spanning tree. If we choose a spanning tree of G for T, then G is clearly embeddable in a $(1, T)$-book. If the graph G is a maximal planar graph, determining $lf(G)$ becomes a problem of finding a spanning tree of G with the least number of endvertices.

For our purposes, two trees that are homeomorphic are considered the same as tree-spines. For example, the simple path of length one P_1 functions the same as the simple path of length $n \geq 1$ P_n as a tree-spine. Figure 3.1 shows all homeomorphically irreducible of trees with $n \leq 6$ endvertices (see Harary and Prins [9]).

Figure 3.1 *Homeomorphically irreducible trees with $n \leq 6$ endvertices.*

The only tree with two endvertices is the path P. It is clear that any sub-hamiltonian planar graph G is embeddable in a $(1, P)$-book, because in a two-page book embedding of G we may use the finite segment of the spine that contains the vertices of G for P. Is the converse true? Is any $(1, P)$-book embeddable graph necessarily subhamiltonian? At first glance, this appears to be true. However, there is a difference in a standard two-page book with an infinite linear spine and a $(1, P)$-book with a spine P of finite length. The definition of an (n, T)-book allows edges on a page to wrap around the ends of the tree-spine T. This introduces the possibility of edges wrapping from one side of P to the other side of P around the endpoints of P. In a standard 2-book, edges cannot wrap from one side of the spine to the other since crossing the spine is not allowed.

We show that this difference is significant. First, we give the following classification of graphs embeddable in a $(1, P)$-book, where P is a line segment in the plane (i.e. P is the path with two endvertices).

Theorem 3.1 *$bt(G, P) = 1$ if and only if G is a subgraph of a planar graph with a Hamiltonian path.*

Proof: Let G be a graph with $bt(G, P) = 1$. Consider a $(1, P)$-book embedding of G. Every vertex of G lies on the line segment P. We form the desired Hamiltonian path by adding edges between consecutive vertices along P if they are not already present. With the added edges, we now have a planar graph with a Hamiltonian path having G as a subgraph.

Conversely, suppose G is a subgraph of a planar graph G' that has a Hamiltonian path. Draw G' in the plane and trace out a Hamiltonian path P' in G'. The path P' is equivalent to the tree with two endvertices P and will act as the spine P in the $(1, P)$-book embedding. Near the spine P we add extra copies of each edge of P', giving us a $(1, P)$-book embedding of G'. Fixing the spine P, we delete the extra vertices and edges of G' to obtain a $(1, P)$-book embedding of G (see Figure 3.2). ∎

Figure 3.2 *A $(1, P)$-book embedding of $St^2(K_3)$.*

The difference between the usual two-page book and a $(1, P)$-book can now be seen. Graphs embeddable in a two-page book are subgraphs of planar Hamiltonian graphs, while graphs embeddable in a $(1, P)$-book need only be subgraphs of planar graphs having Hamiltonian paths. Since there are non-Hamiltonian maximal planar graphs with Hamiltonian paths, there are graphs embeddable in $(1, P)$-books that cannot be embedded in a standard 2-book. As we mentioned in chapter two, the graph $St^2(K_3)$ of Figure 3.2 is the smallest non-Hamiltonian maximal planar graph. Figure 3.2 shows a Hamiltonian path in $St^2(K_3)$ and the corresponding $(1, P)$-book embedding. Hence $lf(St^2(K_3)) = 2$, even though $St^2(K_3)$ needs three pages to be embedded in a standard book, as shown in chapter two.

Although a $(1, P)$-book allows embeddings of more planar graphs than a normal two-page book, there are many planar graphs that are not embeddable in a $(1, P)$-book. To create maximal planar graphs with large leaf numbers, it is clear that we need to have many separating triangles. However, it is not clear exactly how many separating triangles are needed and how they must be placed to increase the leaf number. By Whitney's Theorem, a maximal planar graph G with no separating triangles is Hamiltonian and thus has $lf(G) = 2$. We show that a single separating triangle is not enough to increase the leaf number. In fact, maximal planar graphs with one separating triangle are Hamiltonian. To prove this, we first need the strong form of Whitney's Theorem.

Theorem 3.2 *Let G be a simple connected planar graph without separating triangles drawn in the plane so that every face of G is a triangle, with the possible exception of the outer face. Let R be the circuit of length $k \geq 3$ bordering the outer face. Let u and v be two distinct vertices of R, dividing R into two arcs R_1 and R_2, each including both u and v. Suppose*

(1) The arc R_1 has no chords (i.e. there are no edges of G between vertices of R_1 other than the edges of R), and

(2) Either R_2 has no chords, or else there is a vertex w in R_2 distinct from u and v, dividing R_2 into two arcs R_3 and R_4, each including w, such that R_3 and R_4 have no chords.

Then G has a Hamiltonian path beginning at u and ending at v.

Proof: See Whitney [31]. ■

Now we are ready to prove the following theorem.

Theorem 3.3 *If G is a maximal planar graph with exactly one separating triangle, then G is Hamiltonian.*

Proof: Let G be a maximal planar graph with exactly one separating triangle T. Draw G in the plane. Let G_1 be the graph formed by T and everything lying inside T in the planar embedding of G. Similarly, let G_2 be the graph formed by T and everything lying outside T in the plane. The graphs G_1 and G_2 are maximal planar graphs without separating triangles. We will carefully apply the strong form of Whitney's Theorem to obtain Hamiltonian paths in G_1 and G_2 that we link together to give a Hamiltonian circuit in G.

First, we will obtain a Hamiltonian path in G_2. Consider the triangular face of G_2 bound by the triangle T. We redraw G_2 in the plane so that T bounds the outer face. Now G_2 is a maximal planar graph without separating triangles having the outer circuit T. Let u and v be two distinct vertices of T. Since the outer circuit T is a triangle, there cannot be any chords inside T. Hence, the vertices u and v break T into two chord-free arcs. Thus, by Theorem 3.2, G_2 has a Hamiltonian path beginning at u and ending at v.

Let w be the third vertex of T. The Hamiltonian path that covers G_2 includes w, so it is not sufficient to find a similar path from u to v through w in G_1. In G_1

we seek a path from u to v that avoids w and passes through every other vertex of G_1 exactly once.

Since T is a separating triangle of G, the graph G_1 does not consist of T alone. Since G_1 is maximal, w must be adjacent to vertices other than u and v inside T. Without loss of generality, let $u = v_0, v_1, v_2, \ldots, v_n = v$ be the vertices of G_1 adjacent to w in a clockwise ordering about w. We claim that $u = v_0, v_1, v_2, \ldots, v_n = v$ forms a path in G_1. Since G_1 is a maximal planar graph, if any edge $\{v_i, v_{i+1}\}$ is not present, then w is adjacent to a vertex between v_i and v_{i+1} in the clockwise ordering around w. Hence, all edges $\{v_i, v_{i+1}\}$ are in G_1.

Consider the graph $G_1 - w$ consisting of the n-cycle $C = \{u = v_0, v_1, \ldots, v_n = v\}$ and its interior. We claim that $G_1 - w$ satisfies the conditions of Theorem 3.2. The two chord-free arcs of C are determined by u and v. The arc formed by the edge $\{u, v\}$ clearly has no chords. The other arc formed by the path $u = v_0, v_1, \ldots, v_n = v$ is also chord-free. For if there is an edge $\{v_i, v_j\}$ where v_i and v_j are non-consecutive vertices of C, then the vertices v_i, v_j, and w form a triangle that is not a face in G_1 (i.e. a separating triangle), contrary to the assumption that G has no separating triangles. Now we see that $G_1 - w$ satisfies the conditions of Theorem 3.2. Hence, there is a Hamiltonian path in $G_1 - w$ beginning at u and ending at v (see Figure 3.3).

Figure 3.3

Now we have a u-v path that covers G_2 and one that covers G_1, avoiding w. By connecting the two paths together at u and v, we have a circuit covering every vertex of G exactly once as shown in Figure 3.3. Thus, G is Hamiltonian. ■

In many cases, if G is a maximal planar graph with few separating triangles, then G is Hamiltonian. Exactly how many separating triangles are needed to insure splitting in spanning trees depends on their relative locations in the graph. However, we do have bounds on the leaf numbers of maximal planar graphs. Generally, the leaf number of a maximal planar graph G cannot exceed the number of separating triangles of G. We prove this by induction on the number of separating triangles.

Theorem 3.4 *If G is a maximal planar graph with $n \geq 2$ separating triangles, then $lf(G) \leq n$.*

Proof: Let G be a maximal planar graph, drawn in the plane, with $n \geq 2$ separating triangles. Suppose G has $n = 2$ separating triangles. Let T be a separating triangle of G so that no other separating triangle of G lies inside T in the planar embedding of G. Let G' be the graph formed by deleting the vertices and edges lying inside T. Now T is a triangular face of G', so G' is a maximal planar graph with only one separating triangle. Thus, by Theorem 3.3, G' has a Hamiltonian circuit. This Hamiltonian circuit H includes the three vertices of T. We consider the case in which one (or more) of the edges of T is included in H and the case in which no edge of T is in H.

Case 1: Suppose an edge of T is in H. Let u, v, and w be the three vertices of T in a clockwise ordering. Without loss of generality, assume that the edge $e = \{u, v\}$ is in H. Now let T' be the graph consisting of T and all vertices and edges inside T in the planar embedding of G. Then, using the same technique as in Theorem 3.3, we make a path $u = v_0, v_1, v_2, \ldots, v_p = v$ in T' of vertices adjacent to w in T', in a clockwise ordering about w. This path and the edge e form two chord-free arcs in the outer circuit of $T' - w$. Now, by Theorem 3.2 there is a Hamiltonian path in $T' - w$ beginning at u and ending at v. Hence, we have a path from u to v that covers everything inside T and avoids w. Now we delete the edge e from H and replace it with the u-v path inside T to obtain a Hamiltonian circuit for G. Thus, if an edge of T is in H, then G is Hamiltonian and we see that $lf(G) = 2$.

Case 2: Suppose that H contains no edges of T. Then, in the same way, we form a path from u to v inside T that avoids w. Let x be the last vertex of this path before we reach v. Delete edge $\{x, v\}$ from this path. Now we have a u-x path that covers every vertex in the interior of T and avoids both w and v. Now we join the interior u-x path to H in the following way. Since the edge $e = \{u, v\}$ is not in H, then H can be split into two u-v paths, each having at least one vertex other than u and v. Only one of these paths includes w. Let $u = u_0, u_1, \ldots, u_r = v$ be the u-v path in H that does not contain w. Delete the edge $\{u = u_0, u_1\}$ from this path. Now we follow along H from u_1 to u to form a path that covers G'. We join this path with the u-x path at u to form a Hamiltonian path in G (see Figure 3.4). Hence, $lf(G) = 2$.

Figure 3.4

Now suppose that the theorem holds for planar graphs with $n = 2, 3, \ldots, k-1$ separating triangles. Let G be a planar graph embedded in the plane with $k > 2$ separating triangles. Let T be a separating triangle of G so that no other separating triangles lie within T in the planar embedding of G. If we let G' be the graph having $k-1$ separating triangles formed by deleting all vertices and edges interior to T, then $\mathit{lf}(G') \leq k-1$ by induction. Hence, G' has a spanning tree W' with $k-1$ or fewer leaves.

Again, let u, v, and w be the vertices of T in a clockwise ordering. As in the basis case, if any edge of T is included in W', then we can replace that edge with a path covering everything inside T. If no edge of T is in the spanning tree W', we form a path from u to v that avoids w and covers the interior of T. We delete the last edge of this path before v, giving us a path from u that covers the interior of T and terminates inside T (see Figure 3.5).

Figure 3.5

By joining this path to W' at u we form a spanning tree W of G that has at most one more endvertex than W'. Clearly, by connecting a path to a tree at one point creates another tree. If u was an endvertex of W', then connecting a path to W' at u does not change the number of leaves of the tree. If u was not an endvertex of W', then the addition of a path at u adds one endvertex. Now we see that $lf(G) \leq lf(G') + 1 \leq (k-1) + 1 = k$. Hence, $lf(G) \leq k$. ■

An immediate consequence of Theorem 3.4 is that a maximal planar graph with two or fewer separating triangles is path-Hamiltonian. We also note that at each stage, if any edge of a separating triangle is part of the tree structure of the previous stage, then the leaf number does not increase. Thus, it is likely that the leaf number of a graph is in actuality much lower than the number of separating triangles in the graph.

Using the technique of stellation, we can create maximal planar graphs that do not have Hamiltonian paths. In fact, we will show how to build maximal planar graphs in which the number of endvertices of any spanning tree can be made arbitrarily large.

Define the **inner stellation of the triangle** T_n as follows. T_0 is the triangle and $T_{k+1}, k \geq 0$ is formed by stellating all but the outer face of T_k. Figure 3.6 shows T_0, T_1, T_2, and T_3. The graphs T_2 and T_3 have properties that allow us to build maximal planar graphs with complicated spanning trees.

It is easy to see that beginning at any corner of T_2, one cannot form a path that covers all interior vertices of T_2 without using another corner. Furthermore, if only one additional corner is used, this path must terminate inside T_2. If one begins the path at a corner and ends the path at another corner, then all three corners must be used to cover all vertices inside T_2.

The graph T_3 contains three copies of T_2 joined together as shown by the bold-faced edges in Figure 3.6. We will use the above properties of T_2 to show that we cannot enter T_3 at any corner and form a single path that covers every vertex of T_3 exactly once.

Figure 3.6 *Inner stellations of the triangle.*

Theorem 3.5 *T_3 does not have a Hamiltonian path beginning at a corner of the outer face.*

Proof: By way of contradiction, suppose that T_3 has a Hamiltonian path H beginning at the corner vertex a. Using the vertex labels of Figure 3.6, let A be the copy of T_2 bound by the triangle $\{a, b, x\}$, let B be the copy of T_2 bound by the triangle $\{b, c, x\}$, and let C be the copy of T_2 bound by the triangle $\{c, a, x\}$. Then we have $T_3 = A \cup B \cup C$.

The Hamiltonian path H in T_3 must use the vertices a, b, c, and x. Each of these vertices must have either degree one or degree two in H (with at most two of degree one). Since H begins at the vertex a, then a has degree one in H. We have two possibilities to consider. The first case is if b, c, and x all have degree two in H. The second case is if exactly one of b, c, or x has degree one and the others have degree two in H.

Case 1: Suppose that b, c, and x all have degree two in H. Then the sum of the degrees of a, b, c, and x in H is $1+2+2+2=7$. H must cover each vertex of T_3. Thus, H must cover the vertices inside each of the three copies of T_2. Since H begins at a and since b, c, and x have degree two in H, the other endpoint of H must be in the interior of A, B, or C. So, H must pass through two of these copies of T_2 without terminating inside. By the properties of T_2, we need to use all three corners to cover all vertices inside these two copies of T_2. We also need to use at least two corners of the copy of T_2 containing the second endpoint of H. Hence, the sum of the degrees in H of the corners of A, B, and C must be at least $3+3+2=8$. This is a contradiction of the fact that the sum of the degrees of a, b, c, and x in H is 7.

Case 2: Suppose that one of b, c, and x has degree one in H. Now the sum of the degrees of a, b, c, and x in H is $1+2+2+1=6$. The other vertex of degree one is not inside A, B, or C. Hence, we need to use all three corners to cover the vertices inside each of these three copies of T_2. This gives us at least $3+3+3=9$ for the sum of the degrees of a, b, c, and x in H, exceeding the actual sum of 6.

Thus, T_3 does not have a Hamiltonian path beginning at a corner. ∎

Theorem 3.5 shows us that we cannot enter T_3 from any corner and form a path that hits every vertex. Similarly, if we enter T_3 with through two corners we cannot cover every vertex with these two disjoint paths. Suppose we can cover every vertex of T_3 exactly once with two disjoint paths P_1 and P_2 beginning at a

and b, respectively. Now, we have four vertices of T_3 that have degree one in one of these two paths. As in Theorem 3.5, if either c or x has degree one in either path, we have no chance of covering the vertices inside the three copies of T_2. This leaves the possibility of both paths terminating inside one or two copies of T_2. This would reduce the total number of corners needed to cover the vertices inside the three copies of T_2 by one. But, at the same time, the vertex b has degree one in P_2 instead of two. Hence b can only contribute to covering the vertices inside one, instead of two, copies of T_2. This is the same as Case 1 of Theorem 3.5 with both sums reduced by one. So, we see that two paths through two corners of T_3 cannot cover T_3.

We can continue in the same manner to show that three disjoint paths beginning at a, b, and c cannot include every vertex of T_3 exactly once. This tells us that each copy of T_3 in a graph forces splitting in the underlying tree structure. And since T_3 itself is not path-Hamiltonian, it follows that a planar graph with n copies of T_3 must have $lf(G) \geq n + 2$ (at least two endvertices for the simple path and at least n more endvertices for the copies of T_3).

We can use T_3 to build graphs that cannot be embedded on an n-star. This is done by placing disjoint copies of T_3 in the plane and adding edges to create a single maximal planar graph. Since the copies of T_3 are disjoint and each causes a split in any spanning tree, a single vertex of high degree in a spanning tree will not suffice. Figure 3.7 shows a maximal planar graph with two copies of T_3 that does not have an n-star spanning tree for any n. The leaf number of this graph is six, even though G is not embeddable in the plane with a 6-star spine. The boldfaced edges in Figure 3.7 show a spanning tree of G with six leaves, demonstrating that a graph G with $lf(G) = n$ is not necessarily embeddable on all trees with n endvertices.

Figure 3.7 *A graph that does not have an n-star spanning tree for any n.*

We can use the properties of T_2 to build a family of n-star embeddable graphs. If C_n is the cycle of length n, define S_n to be the third inner stellation of C_n. The graphs $S_3 = T_3$ and S_6 are shown in Figure 3.8. Each graph S_n contains n copies of T_2 situated around a vertex of degree n. The properties of T_2 lead to the result $\mathit{lf}(S_n) = n$. The bold-faced edges of Figure 3.8 represent spanning trees with the minimal number of endvertices. We now have graphs with $\mathit{lf}(G) = n$ for any $n \geq 2$, again showing that we can make planar graphs with arbitrarily large leaf numbers.

Suppose we have an embedding of a graph G in a $(1, T)$-book where T has n leaves. Then G can be embedded in a $(1, T')$-book where T is a sub-tree of T'. We do this by adding the extra vertices and edges of T' next to the spine in the planar $(1, T)$-book embedding of G. Hence, a graph G with $\mathit{lf}(G) = 2$ is embeddable in a $(1, T)$-book, where T is any tree with at least one edge. Since they are Hamiltonian, maximal planar graphs without separating triangles at least have such trivial embeddings on the tree with three endvertices (the $\mathbf{Y}$). We show that a maximal planar graph G without separating triangles has a non-trivial embedding on a **Y**-tree by finding a **Y**-spanning tree of G. In fact, this can be done in such a way that the three leaves of the spanning tree lie on a triangle in G.

Figure 3.8 *The graphs $S_3 = T_3$ and S_6.*

Theorem 3.6 *If G is a maximal planar graph without separating triangles having $n \geq 4$ vertices and if $T = \{a, b, c\}$ is a triangle of G, then G has a $\mathbf{Y}$-spanning tree with endvertices a, b, and c.*

Proof: Let G be a maximal planar graph, drawn in the plane so that the triangle $T = \{a, b, c\}$ bounds the outer face. Without loss of generality, assume that a, b, and c appear in a clockwise ordering in this planar embedding. Since G has at least four vertices and since G is maximal, the vertex a must be adjacent to some vertex x inside T different from b and c. Let $b = v_0, v_1, \ldots, v_{k-1}, v_k = c$ be the vertices of G adjacent to a in a clockwise ordering about a. Now the graph $G - a$, bound by the circuit $\{b = v_0, v_1, \ldots, v_{k-1}, v_k = c\}$ satisfies the conditions of Theorem 3.2 with two chord-free arcs determined by b and c. Hence, $G - a$ has a b-c spanning path. This path passes through the vertex x. By adding the edge $\{a, x\}$ to this path, we form a $\mathbf{Y}$-spanning tree of G with center at x and with endvertices at a, b, and c. ■

There may be stronger theorems giving non-trivial embeddings on $\mathbf{Y}$-trees of maximal planar graphs without separating triangles. For example, we may be able to choose any triangle of the graph for the set of endvertices and any other vertex as the center of the $\mathbf{Y}$-tree. In hopes of proving this, we attempted an inductive argument similar to that of the strong form of Whitney's Theorem. We start with a graph G in which every face is a triangle, except possibly for the outer face, bound by an n-cycle. We hoped to show that if the outer n-cycle has three chord-free arcs determined by vertices a, b, and c, and if x is a vertex of the graph inside the outer n-cycle, then G has a $\mathbf{Y}$-spanning tree with center at x and endvertices a, b, and c. Then the result would follow for maximal planar graphs, taking any triangle for the outer n-cycle. The graph in Figure 3.9 shows that this proof technique will not work. This graph has three chord-free arcs determined by a, b, and c. However, one cannot form a **Y**-spanning tree with center at x and leaves at a, b, and c.

Figure 3.9

Although this technique did not lead to a successful proof, the example of Figure 3.9 is not maximal. Thus, it is not a counter-example to the original claim about maximal planar graphs without separating triangles. The proof of Theorem 3.6 shows that any vertex inside the selected outer triangle T that is distance one from any vertex of T can be picked for the center of a **Y**-spanning tree. However, it is still uncertain whether any other vertex inside T will work as the center. In fact, we may be able to choose **any** four vertices in a maximal planar graph without separating triangles and form a **Y**-spanning tree having three of these vertices as leaves and the other as the center. We have been unable to find examples of such graphs for which this is not possible.

Theorem 3.6 gives us more than non-trivial embeddings of maximal planar graphs without separating triangles in **Y**-tree books. It demonstrates that a maximal planar graph G without separating triangles has a **Y**-spanning tree with all leaves on one face. If drawn so that this face is the outer face, we have a **Y**-tree embedding of G in which the edges of G do not wrap around the ends of the **Y**-tree. This leads to a modification of the tree-book embedding problem in which edges cannot wrap around the ends of the spine.

To realize this modification, we can extend infinite paths from the endvertices of the tree-spine. Let T be a tree drawn in the plane. The **extended T-spine T'** is the spine formed by drawing infinite rays in the plane, called **extended ends**, extending from the endvertices of T (without crossing any part of T or each other). If a tree has n leaves, this process will split the plane into n regions. For example, the extended P-spine (where P is the tree with two endvertices) splits the plane into two regions. The problem of embedding graphs on a page with an extended P-spine is the same as standard two-page book embedding problem. Since there are graphs that are embeddable in a $(1, P)$-book but are not subhamiltonian, it is clear that the infinite extension of the P-spine makes a difference in the embedding problem.

It does not take long to see that extending the ends also makes a difference with other spines as well. Let $\mathbf{Y}'$ be the structure formed by extending the ends of the $\mathbf{Y}$-tree. The graph T_3 has a one-page $\mathbf{Y}$-book embedding (see Figure 3.8). In this embedding, the three ends of the $\mathbf{Y}$-spanning tree all lie on different faces in the graph. The properties of T_2 force each of the ends of the $\mathbf{Y}$-tree to terminate inside different copies of T_2. Hence, T_3 does not have a one-page $\mathbf{Y}'$-book embedding.

If the extended spine is formed from an original spine with four or more vertices, then some edges between vertices of the graph may not even be possible to draw without crossing the spine. For example, consider the spine formed by extending the ends of the 4-star. We may represent such a page by two perpendicular lines that cross in the plane (the x and y axes), dividing the plane into four regions (quadrants). If one of the lines has two vertices placed on opposite sides of the intersection point, no edge can join them without crossing one of the lines. Hence, with a given placement of vertices, there are certain edges that cannot be embedded on the extended tree page. In a standard tree book any single edge can be drawn on a page regardless of the vertex placement.

Not only does the extended spine limit which edges can appear, but it limits the number of edges that can be placed on a page. Let T be a tree with k leaves drawn in the plane and let T' be the corresponding extended T-spine. We will say that the planar graph G has a non-trivial one-page T'-book embedding if G can be embedded in a one-page T'-book with at least one vertex on each of the k extended ends. We will show that when $k \geq 4$, a one-page T' book generally allows fewer edges for a non-trivially embedded n vertex graph than a $(1, T)$-book.

From Euler's Formula for planar graphs, it follows that a simple planar graph with n vertices has at most $3n - 6$ edges (corresponding to a complete triangulation). This maximum can be reached in a standard two-page book since we can have up to n edges for a complete Hamiltonian circuit, up to $n - 3$ edges to triangulate the inside of this circuit, and up to $n - 3$ edges to triangulate the outside of this circuit. For any tree T, we can form a maximal planar graph having T as a spanning tree by triangulating T. Thus, the bound can be reached in $(1, T)$-books. Since a one-page book with an extended P-spine is the same as a standard 2-book, the bound of $3n - 6$ edges can be reached if G is a maximal subhamiltonian planar graph with n vertices. We also showed that maximal planar graphs without separating triangles have non-trivial embeddings on one-page books with extended $\mathbf{Y}$-spines, giving us a set of $\mathbf{Y}'$-book embeddable graphs that attain the bound.

We now consider extended spines T', where T is a tree with $k \geq 4$ leaves. Recall that the extended spine T' divides the plane into k regions. Let G be a planar graph with n vertices that has a non-trivial one-page T'-book embedding. In such an embedding, the $n \geq k$ vertices of G are placed on T' so that at least one vertex lies on each of the k extended ends. We will show that G must have fewer than $3n - 6$ edges. First, note that one edge can be added across the top of each of the k regions joining the last vertex of each the two extended edges that form the region. Since these k edges can be added to any graph in a non-trivial T'-book embedding without crossing any other edge of the graph, we may as well assume that these edges are present in G. The union of these k edges forms a k-cycle so that every edge of G is either in this k-cycle or lies within this k-cycle in a planar embedding of G. Now we have a k-face of G with $k \geq 4$. Thus, G is not maximal, so G has fewer than $3n-6$ edges. Since we cannot triangulate this outer k-cycle, in general the number of edges of G cannot exceed $(3n-6)-(k-3) = 3n-k-3$.

We will show that this bound is the best possible bound by giving a family of graphs B_k so that for $k \geq 4$, B_k has an extended k-star embedding and satisfies the edge bound. Let $B_k, k \geq 2$ be the graph formed as follows. First, stellate a $2k$-cycle, forming $2k$ triangles. Now stellate every other triangular face twice, forming k copies of T_2. Finally, we begin at any vertex v of the $2k$-cycle and make a path of length k by connecting vertices at distance two around the $2k$-cycle until we return to v (see Figure 3.10). The bold-faced edges of Figure 3.10 show the extended k-star that acts as the spine. It is easy to see that we actually reach the bound since every face but the outer k-face is a triangle. Combining these bounds with those for $\mathbf{Y}'$ embeddable graphs, the following theorem is clear.

Theorem 3.7 *If T is a tree with $k \geq 3$ leaves and if T' is the extended tree spine corresponding to T, then any graph G on n vertices that admits a non-trivial T'-book embedding has at most $3n-k-3$ edges.* ∎

It is clear that the set of graphs that admit a $(1,T)$-book embedding includes all graphs embeddable in a one-page tree book with an extended T'-spine. In fact, we can do a lot better than this. If T' is an extended spine formed from a tree T with $k = 2, 3$, or 4 endvertices, then any graph embeddable on a one-page T' tree book is embeddable on a $(1,P)$-book. Similarly, if $k = 5$ or 6, then any

Figure 3.10 *The graph B_3.*

graph embeddable in a book with the extended spine T' is embeddable in a book with a **Y**-tree spine. This is done by pairing consecutive extended ends of T' in a clockwise fashion, adding the edge across the top of the region determined by each pair. The new spine follows T' to this added edge, goes across the edge and then follows T' back to the point at which the next extended edge connects to T'. Figure 3.11 shows how to make this transformation for all spines formed by extending trees with $k = 2, 3, 4, 5$, and 6 leaves.

Figure 3.11 *Transformations of extended spines of trees with $k = 2, 3, 4, 5,$ and 6 leaves.*

Continuing this pairing process leads to the following result.

Theorem 3.8 *Let T be a tree with k endvertices and let T' be the extended spine corresponding to T. If G is embeddable in an n-page tree book with extended spine T', then G is embeddable in an (n, S)-book where S is a tree with $\lceil k/2 \rceil$ endvertices.* ∎

In this chapter, we explored the problem of embedding graphs in books with spines that are trees. We focused our attention on planar graphs. By restricting ourselves to single page embeddings, we saw that a tree-spine with the fewest endvertices needed to embed a particular graph can be very complicated. In the next chapter, we go back to a simple spine. We consider books with linear (or circular) spines but with modified pages.

Chapter 4

EMBEDDING GRAPHS IN GENERALIZED BOOKS

In Chapter 3 we generalized the book embedding problem by modifying the spine. Now we return to the notion of the spine as a straight line, or circle, in 3-space. This time we will instead modify the pages.

The first modified pages we consider are ones that bend and reconnect at the spine. These **cylinder pages** can be viewed as the shells of concentric cylinders, connected together at the spine which is a straight line segment between the two bases of each cylinder. To embed a graph in a **cylinder book**, the vertices are all placed on the spine and the edges on the cylinder pages so that no two edges on one page cross. We define the **cylinder thickness** $ct(G)$ to be the smallest number of cylinder pages needed to embed the graph G in a cylinder book. If we cut a one-page cylinder book at the spine and flatten it out, we get a planar structure. This can be realized by making two parallel copies of the spine in the plane. Then all edges that fit on a cylinder page can be drawn in the space between the two copies of the spine. Figure 4.1 shows a one-page cylinder book embedding of $St^2(K_3)$.

If there are no edges between the two copies of the spine, then we have a standard two-page book. Thus, the set of graphs that are embeddable in a one-page cylinder book includes all subhamiltonian graphs. But, we know that $St^2(K_3)$ is not subhamiltonian. Hence, a one-page cylinder book allows embeddings of more

Figure 4.1 *One-page cylinder book embedding of $St^2(K_3)$.*

planar graphs than the standard two-page book. How much better is a one-page cylinder book? The following theorem helps answer this question.

Theorem 4.1 *Let G be a graph. Then $ct(G) = 1$ if and only if $lf(G) = 2$.*

Proof: Suppose that $ct(G) = 1$. Then G can be embedded in a one-page cylinder book. Now we realize this embedding in the plane cutting the cylinder page along the spine. The edges of G lie in the plane between two parallel copies of the spine. We now bend the edges of G and re-join the two copies of the spine together in the plane (see Figure 4.2). Now we have a planar embedding of G so that every vertex of G lies on a path (i.e. a tree with two leaves). Hence, $lf(G) = 2$.

Conversely, suppose that $lf(G) = 2$. Then the vertices of G can be placed on a straight line segment P in the plane so that every edge of G lies either above P, below P, or wraps around an endpoint of P. Without violating planarity, the edges that wrap around the endpoints of P can be arranged so that they all wrap around the right endpoint of P. Now we reverse the process and cut along P from left to right to make two copies of P. Rotate the bottom copy of P counter-clockwise about its right endpoint until the two copies of P are parallel in the plane. The edges of G now lie in the plane between the copies of P (see Figure 4.2). We reconnect the spine to form the cylinder embedding. Thus, $ct(G) = 1$. ■

Figure 4.2

So, if P is the tree with two endvertices, then Theorem 4.1 gives us that $bt(P, G) = ct(G)$. In other words, the number of pages needed to embed a graph G in a tree-book with a path spine is the same as the number of cylinder pages needed to embed G in a cylinder book. The one-page cylinder book and the one-page tree book with path spine P are similar to a standard two-page book in that they only admit embeddings of planar graphs. Both structures differ from a standard two-page book in that they allow the wrapping of edges. In the tree-book, edges can pass between opposite sides of the spine by bending around the ends of the tree. In the cylinder book, edges can wrap from one side of the spine to the other over the surface of the cylinder.

What if we allow edges to wrap in two directions? The second page modification we consider is a torus page. The spine of a **torus book** is a ring on a torus. Multiple concentric **torus pages** are joined together at this common spine. Again, when embedding a graph in a torus book, the vertices are placed on the spine and the edges on the torus pages without crossing. The **torus thickness** $t(G)$ of a graph G is the least number of torus pages needed to embed G in a torus book.

We also have a simple way of realizing a single torus page. We cut the circular spine to make a line segment. Now we draw two parallel copies of the spine in the plane forming the top and bottom sides of a rectangle. This cylinder page is transformed into a torus page by identifying the two vertical sides of this rectangle (see Figure 4.3). Edges may pass through a vertical side of the rectangle and re-enter at the same point on the opposite side. This horizontal wrapping of edges allows the embedding of many graphs in a one-page torus book that do not admit embeddings in a one-page cylinder book. Figure 4.3 shows a one-page torus book embedding of the complete graph K_7.

The graph K_7 is a non-planar graph by a theorem of Kuratowski (see [1], p. 53). Thus, K_7 cannot be embedded either on a one-page cylinder book or on a standard two-page book. By the construction of a torus book it is clear that any graph embeddable on a one-page cylinder book is also embeddable in a one-page torus book. How much better is the torus page? To help answer this question, we will look at the number of edges allowed on a torus page with n vertices on the

Figure 4.3 *One-page torus book embedding of K_7.*

spine. Additionally, we will compare the one-page torus book and the standard k-page book.

First, we will examine the standard k-book. Recall that an n-vertex graph embeddable in a standard two-page book has at most $n+2(n-3) = 3n-6$ edges. Similarly, a k-page embeddable graph with n vertices can have at most $n+k(n-3)$ distinct edges. We again can have n edges for the outer cycle and up to $n-3$ non-cycle edges on each of the k pages. This gives the following bound on the book thickness given by Bernhart and Kainen [2].

Theorem 4.2 *Let G be a graph with $n \geq 4$ vertices and q edges. Then*

$$bt(G) \geq \frac{q-n}{n-3}.$$

■

We can find bounds for the torus thickness of a graph with fixed numbers of vertices and edges in a similar way.

Theorem 4.3 *Let G be a graph with n vertices and q edges. Then*

$$t(G) \geq \frac{q-n}{2n}.$$

Proof: Suppose that G is a graph with n vertices and q edges. Consider an embedding of G in a $k = t(G)$-page torus book. The n edges between consecutive vertices of G on the spine can be added on any torus page. So, G can have up to n edges along the spine. Using the above representation of the torus, we have two possibilities. Either no edge extends from the upper copy of the spine to the lower copy of the spine, or there is at least one edge that begins at the top copy of the spine and ends at the lower copy of the spine.

Case 1: Suppose there are no edges extending between the two copies of the spine. Then we have a spine with each edge appearing on only one of the two sides of the spine. This structure is equivalent to a standard two-page book. Hence, if no edge has its ends on opposite sides of the spine, then there are $2(n-3) = 2n-6$ edges that can be added to any torus page in addition to the n for the spine. Thus, there are at most $n + 2k(n-3)$ possible edges in an n-vertex graph that admits a k-page torus book embedding with the restriction that no edges extend from one side of the spine to the other.

Case 2: Suppose that an edge e has ends on opposite sides of the spine on a torus page in a torus book embedding of G. We now use the previous representation of a page of a torus book with two parallel copies of the spine. Label the vertices of G $v_1, v_2, \ldots, v_n$ along the spine. The edge e begins at vertex v_i in the top copy of the spine and ends at vertex v_j of the bottom copy of the spine (see Figure 4.4).

Figure 4.4

Now any edge that can be placed on this page is either one of the n edges along the spine, the edge e, or lies within the $2n + 2$-cycle bound by the two copies of the spine and e. Following the arrowed edges of Figure 4.4, this cycle is given by $\{v_i, v_{i+1}, \ldots, v_n, v_1, v_2, \ldots, v_{i-1}, v_i,$ (from left to right across the top) $v_j, v_{j-1}, \ldots, v_2, v_1, v_n, \ldots, v_{j+1}, v_j,$ (from right to left across the bottom) $\}$. There are $2n+2-3 = 2n-1$ possible edges that can be drawn in a complete triangulation of this $2n$+2-cycle. Hence, on a single torus page we can have up to $n+1+(2n-1) = 3n$ edges if there are edges joining opposite sides of the spine. Thus, in a k-page book, there are at most $n + 2kn = (2k+1)n$ edges for an n-vertex graph if edges wrap from one side of the spine to the other.

We see that more edges are allowed on a torus page if some edges wrap around the spine as in Case 2. Hence, if G is a graph with n vertices and q edges embedded in a k-page torus book, it follows that $q \leq (2k+1)n$. Thus, $k = bt(G) \geq \frac{q-n}{2n}$. ■

Now we have lower bounds on the book thickness and torus book thickness of graphs with fixed numbers of edges and vertices. We will show that for both types of books, equality holds in the case where G is the complete graph K_n. The following theorem of Bernhart and Kainen [2] and Ollmann [24] gives the (optimal) book thickness of K_n.

Theorem 4.4 *If $n \geq 4$, then $bt(K_n) = \lceil n/2 \rceil$.*

Proof: Let $n \geq 4$. First, we show that $bt(K_n) \geq \lceil n/2 \rceil$. The graph K_n has n vertices and $\binom{n}{2}$ edges. Now by Theorem 4.2, we have that

$$bt(K_n) \geq \frac{\binom{n}{2} - n}{n-3} = \frac{n(n-1)/2 - n}{n-3} = n/2.$$

Since the book thickness of a graph must be an integer, it follows that $bt(K_n) \geq \lceil n/2 \rceil$.

To obtain the other inequality, we will assume that n is even. Suppose that $n = 2m$. We will show that $bt(K_{2m}) \leq 2m/2 = m$. The result for odd n will follow from the fact that K_{2m-1} is a subgraph of K_{2m}. The m pages of the book are formed by rotating the triangulated $2m$-gon of Figure 4.5 through m successive positions.

Figure 4.5 *Triangulation of the $2m$-gon.*

A triangulation of the $2m$-circuit has $2m-3$ edges. It is easy to see that each inner diagonal of this $2m$-circuit cannot appear in more than one of the m rotations. We get $2m$ edges for the outer circuit and $m(2m-3)$ edges for the m triangulations for a total of $2m + m(2m-3) = 2m^2 - m = \binom{2m}{2}$ distinct edges. Hence, all $\binom{2m}{2}$ edges of K_{2m} are accounted for and we have the desired result. ■

In the case of the torus book, we also achieve the lower bound of Theorem 4.3 with the graph K_n.

Theorem 4.5 *If $n \geq 1$, then $t(K_n) = \lfloor n/4 \rfloor$.*

Proof: Let $n \geq 1$. We apply Theorem 4.3 with n vertices and $\binom{n}{2}$ edges. This gives us that

$$t(K_n) \geq \frac{\binom{n}{2} - n}{2n} = (n-3)/4.$$

Since $t(K_n)$ is an integer, it follows that $t(K_n) \geq \lfloor n/4 \rfloor$.

To show that $t(K_n) \leq \lfloor n/4 \rfloor$, we will assume that $n = 4m + 3$. Since K_{4m}, K_{4m+1}, and K_{4m+2} are all subgraphs of K_{4m+3}, the result will hold for these graphs as well. Label the vertices $v_1, v_2, \ldots, v_{n-1}, v_n$ along the spine. The n edges $\{v_i, v_{i+1}\}$ and $\{v_n, v_1\}$ can all be placed along the spine on any page of the torus book. We will show how to embed the rest of the

$$\binom{4m+3}{2} - (4m+3) = 2m(4m+3)$$

edges of K_{4m+3} on m pages to obtain the desired result.

Consider the circuit $C = \{v_1, v_2, \ldots, v_n\}$. We say that vertices v_i and v_j are **at distance** $k \leq n-1$ **along** C if there is a clockwise simple path of length k in C between v_i and v_j. For example, v_1 and v_2 are at distance one along C and v_1 and v_n are at distance $n-1$ along C. To cover every edge of K_n, we need to ensure that each of the edges joining each vertex v_i to the vertices at distance $1, 2, \ldots, n-1$ from v_i along C appears on a page of the m-page torus book. We will place the remaining $2m(4m+3)$ edges on the m pages so that edges joining vertices at distance $2k, 2k+1, n-2k-1$, and $n-2k$ appear on page k for $k = 1, 2, \ldots, m$. As k ranges over the integers $1, 2, \ldots, m$, $2k$ and $2k+1$ take on the values $2, 3, \ldots, 2m$, and $2m+1$. Similarly, $n-2k-1$ and $n-2k$ take on the values $2m+2, 2m+3, \ldots, 4m$, and $4m+1 = n-2$. Hence, all edges for vertices at distances in the range $2, 3, \ldots, n-2$ appear on the m pages. The edges on page k do not cross as shown by the one-page embedding of K_7 in Figure 4.3.

Each of the $n = 4m+3$ vertices has degree four on each page (not including the edges of the n-cycle along the spine). Since each edge has two distinct endvertices, there are a total of $4(4m+3)/2 = 2(4m+3)$ edges on each of the m pages. Along with the $n = 4n+3$ vertices along the spine, we have accounted for all

$$n + 2mn = (2m+1)(4m+3) = \binom{4m+3}{2}$$

edges of K_{4m+3}. ■

Theorem 4.5 gives a method for attaining one-page torus book embeddings of K_n for $n \leq 7$. The graph K_7 is the largest complete graph that is embeddable on a torus, without any restrictions on the placement of vertices. Now we see that K_7 fits on the torus with all vertices restricted to a line. Theorem 4.4 shows that K_7 requires $\lceil 7/2 \rceil = 4$ pages for an embedding in a standard book. Hence, K_7 is a graph that admits an embedding in a one-page torus book but is not embeddable in a standard 3-book. This leads to the following question: Does any graph G with $bt(G) = 3$ have $t(G) > 1$?

To answer this question, we consider the graph on 10 vertices formed by an outer 10-cycle and three rotations of the triangulation of Figure 4.5. Each triangulation fits on a single page, giving a three-page standard book embedding of the graph. This graph has 10 edges for the outer cycle and $3(10-3) = 21$ edges for the three triangulations for a total of 31 edges and 10 vertices. From Theorem 4.3, a graph G with 31 edges and 10 vertices must have $t(G) \geq (31-10)/2(10) = 21/20 > 1$. Since $t(G)$ is an integer, it follows that $t(G) \geq 2$. Thus, a standard 3-book and a single torus page are generally not comparable.

A standard three-page book admits at most $n + 3(n-3) = 4n - 9$ edges for an n-vertex graph. The one-page torus book allows at most $n + 2n = 3n$ edges of an n-vertex graph. So, for a graph with $n \leq 8$ vertices, the one-page torus book allows more edges. When a graph has $n \geq 10$ vertices, the standard three-page book allows more edges. The number of allowed edges is equal for a graph with nine vertices. The difference between the standard 3-book and the one-page torus book goes beyond the number of edges allowed. We will show that the type of edges allowed varies with the type of book.

Chung, Leighton, and Rosenberg [6] define the **depth-n sum of triangles graph** D_n as the graph with $3n$ vertices $\{a_i, b_i, c_i \mid 1 \leq i \leq n\}$ and edges $\{a_i, b_i\}$, $\{b_i, c_i\}$, and $\{a_i, c_i\}$ for $i = 1, 2, \ldots, n$. If each of the triangles is placed separately along the spine, only one standard page is needed to embed D_n. Now suppose that we add the restriction that the vertices must appear in the order $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n$ along the spine. It is easy to see that any two edges $\{a_i, b_i\}, \{a_j, b_j\}$ $(i \neq j)$ must cross with this ordering of the vertices. Hence, at least n pages are needed to embed D_n in a standard book with the given vertex-ordering. Since each of the n triangles are embeddable in a single page, it follows that n pages are also sufficient.

Now consider embedding D_n in a torus book or a cylinder book with this vertex-ordering. In the cylinder book, we can handle all of the edges $\{a_i, b_i\}$ and $\{b_i, c_i\}$ on a single cylinder page, while all edges $\{a_i, c_i\}$ fit on a second page as shown in Figure 4.6. By wrapping the edges $\{a_i, c_i\}$ around the side, we can see that only one torus page is needed to embed D_n with the above vertex-ordering. Hence, with a given vertex-ordering of a graph G, the thickness of a book needed to embed G varies greatly depending on the type of book we choose.

Figure 4.6 *Two-page cylinder book embedding of D_5.*

It is interesting to note that we can add the edges $\{a_i, a_{i+1}\}$ for $i = 1, 2, \ldots, n-1$, $\{c_j, c_{j+1}\}$ for $j = 1, 2, \ldots, n-1$, $\{c_1, a_n\}$, and $\{a_1, c_n\}$ to the above one-page torus book embedding of D_n without causing edge crossings. When $n \geq 3$, the graph D'_n formed by adding these edges to D_n is a non-planar graph since the two vertex sets $\{a_1, c_2, a_3\}$ and $\{c_1, a_2, c_3\}$ determine a subgraph of D'_n that is homeomorphic with a $K_{3,3}$. Hence, when $n \geq 3$, the non-planar graph D'_n is embeddable in a one-page torus book with the above prespecified vertex-ordering. However, since it is not planar, $D'_n (n \geq 3)$ requires at least three pages for an embedding in a standard book with any ordering of the vertices.

Since the torus page is a non-planar structure, it is also useful for embedding other non-planar graphs. We have already seen that a one-page torus book admits embeddings of the non-planar graphs K_5, K_6, and K_7. Another famous non-planar graph is the Petersen graph (see the first graph of Figure 4.7). A generalization

Figure 4.7 *Generalized Petersen graphs.*

of the Petersen graph includes several non-planar graphs that are embeddable on a single torus page. The **generalized Petersen graph** $P(n,k),(n \geq 3, k \leq n)$ is formed by first creating an outer n-cycle. Then n spokes are added to this cycle by attaching single edges to each of the n vertices. Finally, we add edges joining every kth spoke, resulting in a graph with $2n$ vertices and between $2n$ and $3n$ edges. If $k=0$ or $k=n$, there are no edges added between spokes. If $k=n/2$, there are $n/2$ edges added between spokes. Otherwise, there are n edges added between spokes, with these n edges forming a single cycle when n and k are relatively prime.

The standard Petersen graph is $P(5,2)$. The graphs $P(6,0)$, $P(6,1)$, $P(6,2)$, $P(6,3)$, and $P(9,2)$ are also shown in Figure 4.7. When $k=2$ and $n \geq 5$ is odd, Kuratowski's Theorem can be used to show that the graph $P(n,k)$ is not planar. However, these graphs are all embeddable in a one-page torus book. Observing that $P(n,k) = P(n,n-k)$, we will show that when either $k \leq 2$ or $k \geq n-2$, the generalized Petersen graph is embeddable on a single torus page.

Theorem 4.6 $t(P(n,k)) = 1$ *when either* $0 \leq k \leq 2$ *or* $n-2 \leq k \leq n$.

Proof: Let $n \geq 3$. We will consider three cases:

Case 1: Let $k=0$ or n. The graph $P(n,0) = P(n,n)$ consists only of an outer n-cycle with n spokes. This graph admits a standard one-page book embedding as shown in Figure 4.8. Thus, $P(n,k)$ is embeddable in a one-page torus book when $k=0$ or n.

Case 2: Let $k=1$ or $n-1$. The graph $P(n,1) = P(n,n-1)$ consists of two circuits, the outer n-cycle $\{v_1, v_2, \ldots v_n\}$ and the inner n-cycle $\{u_1, u_2, \ldots u_n\}$ along with the edges $\{v_i, u_i\}$. The $2n$-cycle $\{v_1, v_2, \ldots v_n, u_n, u_{n-1}, \ldots u_2, u_1\}$ forms a Hamiltonian cycle in $P(n,1) = P(n,n-1)$. Since the graph is planar, it is two-page embeddable and thus embeddable on a single torus page (see Figure 4.9).

Figure 4.8 *One-page book embedding of* $P(6,0) = P(6,6)$.

Figure 4.9 *One-page torus book embedding of* $P(6,1) = P(6,5)$.

Case 3: Let $k = 2$ or $n - 2$. If n is even, say $n = 2m$, the edges connecting the spokes of $P(n, 2) = P(n, n-2)$ form two m-cycles. We can draw one of these cycles inside the outer n-cycle and the other outside the outer cycle without edge crossings. Hence, we have a planar drawing of $P(2m, 2) = P(2m, 2m - 2)$ so that there are no triangles other than faces. Thus, by Theorem 2.9, the graph is subhamiltonian and, hence, is embeddable on one torus page.

Now suppose that $n = 2m + 1$. Since n is odd, n and 2 are relatively prime. Hence, the edges connecting the spokes of $P(n, 2) = P(n, n - 2)$ form an inner n-cycle. Label the outer n-cycle $\{v_1, v_2, \ldots, v_n\}$ in clockwise order. Now label the inner n-cycle $\{u_1, u_2, \ldots, u_n\}$ in clockwise order, where u_1 is adjacent to v_n. Now line the vertices along the spine $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$ (see Figure 4.10).

Figure 4.10 *One-page torus book embedding of* $P(9, 2)$*.*

Adding edges between consecutive vertices along the spine covers $2n-1$ of the $3n$ edges of the graph. We still need to place the other $n-1$ spokes (all but $\{u_1, v_n\}$), the edge $\{v_1, v_n\}$, and the edge $\{u_1, u_n\}$ on the torus page.

The remaining $n-1 = (2m+1)-1 = 2m$ spokes fall into two sets. The first set consists of the m edges $\{v_{2i-1}, u_{m+i+1}\}$ $(1 \leq i \leq m)$ and the second group has the m edges $\{v_{2i}, u_{i+1}\}(1 \leq i \leq m)$. Place the first group of edges so that the first vertex (i.e. the v vertex) of each edge is on the upper copy of the spine and the second vertex is on the lower copy on the torus page. Next, place the second group of edges so that the u vertex is on the upper copy of the spine and the v vertex is on the lower copy of the spine. By wrapping edges from left to right, the two sets of edges can be placed in this manner without edge crossings as shown in Figure 4.10.

The two edges $\{v_1, v_n\}$ and $\{u_1, u_n\}$, represented by dotted lines in Figure 4.10, can be placed between the two sets of spoke edges to complete the one-page torus book embedding of $P(2m+1, 2)$. ■

Theorem 4.6 is significant when $n \geq 5$ and n is odd. For these values of n the graph $P(n, 2)$ contains a subgraph that is homeomorphic with the non-planar graph $K_{3,3}$. Hence, these graphs are not planar and do not have standard 2-book embeddings. When $n \geq 6$ is even and $k = n/2$, $P(n, k)$ is also non-planar. The next theorem shows that these graphs have one-page torus book embeddings as well.

Theorem 4.7 *If $n = 2m$, then $t(P(n, m)) = 1$.*

Proof: Let $m \geq 2$ and consider the graph $P(2m, m)$ drawn as in Figure 4.11. We again give a clockwise labeling of the outer circuit $\{v_1, v_2, \ldots, v_n\}$. Now label the ends of the spokes $u_1, u_2, \ldots, u_{2m}$ with a clockwise ordering where u_1 is adjacent to v_1. The vertices will then be lined up on the spine in the order $v_1, v_2, \ldots, v_{2m}, u_1, u_{m+1}, u_2, u_{m+2}, \ldots, u_k, u_{m+k}, \ldots, u_m, u_{2m}$. All but one of the $2m$ edges of the outer circuit can be placed along the spine. We can also place the m edges $\{u_i, u_{m+i}\}$ $(1 \leq i \leq m)$ along the spine. This accounts for $3m-1$ of the $5m$ edges of $P(2m, 2)$. We still need to place the $2m$ spokes and the edge $\{v_1, v_{2m}\}$.

Figure 4.11 *One-page torus book embedding of* $P(6, 3)$.

As shown in Figure 4.11, the m edges $\{v_i, u_i\}$ $(1 \leq i \leq m)$ can be placed without crossing by using the v_i of the top copy of the spine and u_i of the bottom copy of the spine. Similarly, for $m+1 \leq i \leq 2m$, the m edges $\{v_i, u_i\}$ can be placed with the u_is on the upper copy of the spine and the v_is on the lower copy of the spine. Finally, the edge $\{v_1, v_{2m}\}$ can be placed between these two sets of edges on this single torus page. Now all $5m$ edges have been placed on a single torus page without crossing. ■

The standard book, the cylinder book, and the torus book are all orientable surfaces. We can also consider non-orientable book structures by putting a twist on the page. We form a **Möbius page** by cutting a cylinder page along the spine, twisting the page, and joining it back together at the spine. This can be represented by placing two parallel copies of the spine horizontally in the plane. The top copy of the spine is given the usual ordering and the lower copy is a horizontal reflection of the top copy, simulating the one-sided surface. This Möbius page clearly allows the embedding of any graph embeddable in a standard 2-book. The Möbius page embedding of the non-planar graph K_5 in Figure 4.12 shows that this non-orientable page is better than a standard two-page book. Comparing a Möbius book with a torus book or cylinder book is difficult since orientable genus and non-orientable genus are generally not comparable.

Figure 4.12 *One-page Möbius book embedding of K_5.*

In our version of the Möbius page, the spine cuts across the Möbius strip. Another possibility, offered by Kainen [16], is that the spine could follow along the edge of the Möbius band. If we draw the spine as a line in the plane, the edges can be drawn below the spine, wrapping around the sides so that edges exiting on one side return with their order reversed. This difference is significant as illustrated by the depth-n sum of triangles graph D_n. If we again restrict the vertex-ordering along the spine to be $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n$, the number of the first type of Möbius pages required can be made arbitrarily large by increasing n. However, in the second version of the Möbius book, only three pages are needed to embed D_n with this vertex-ordering. The reversal of the edge ordering makes it possible to place the edges $\{a_i, b_i\}$ on a single page without crossing. The sets of edges $\{b_i, c_i\}$ and $\{a_i, c_i\}$ can each be assigned to their own page in a similar way.

When G is a Hamiltonian graph that has an embedding in a standard two-page book we can use any Hamiltonian circuit of G for the ordering along the spine in a two-page embedding of G. For non-planar Hamiltonian graphs, it is often advantageous to choose a Hamiltonian circuit for the vertex-ordering on the spine. This allows the placement of $n = |V(G)|$ edges of G on any page without conflict. However, in the case of non-planar graphs, not every Hamiltonian circuit will give us an optimal embedding.

Chung, Leighton, and Rosenberg [6] present a family of graphs that illustrate the importance of choosing a good Hamiltonian circuit for the ordering on the spine. They define the **depth-n pinwheel graph** PW_n on the $2n$ vertices $\{a_i, b_i \mid 1 \leq i \leq n\}$. The edges of PW_n are given by $\{a_i, b_i\}$ $(1 \leq i \leq n)$, $\{a_i, b_{n-i+1}\}$ $(1 \leq i \leq n)$, $\{a_i, a_{i+1}\}$ $(1 \leq i < n)$, and $\{b_i, b_{i+i}\}$ $(1 \leq i < n)$. The depth-6 pinwheel graph PW_6 is shown in Figure 4.13.

Figure 4.13 *The pinwheel graph PW_6.*

When $n \geq 3$, PW_n is not planar. Hence, at least three pages are needed to embed PW_n in a standard book. Chung, Leighton, and Rosenberg show that $bt(PW_n) = 3$ when $n \geq 3$ by finding a particular Hamiltonian circuit of PW_n for the ordering along the spine. This circuit is not an obvious choice. If we choose the obvious outer Hamiltonian circuit $\{a_1, a_2, \ldots, a_n, b_n, b_{n-1}, \ldots, b_1\}$ for the ordering on the spine, the number of pages needed grows on the order of n.

Instead of searching for another ordering along the spine, we will use the obvious ordering and look for a new type of book that will minimize the number of pages needed to embed PW_n. As illustrated in Figure 4.14, two of the second type of Möbius pages will work with this ordering. Figure 4.15 shows that two torus pages are also sufficient for PW_n with this vertex-ordering.

Finally, we offer a combination of the Möbius page and the torus page that reduces the number of pages needed to one. We form the **Klein bottle page** by

Figure 4.14 *Two-page Möbius book embedding of PW_6.*

Figure 4.15 *Two-page torus book embedding of PW_6.*

drawing two parallel copies of the spine in the plane (with the vertices on each copy ordered in the same way from left to right). Edges that exit one side come back on the other side with the ordering reversed. Figure 4.16 shows how PW_n can be embedded in a single Klein bottle page.

The standard book embedding problem essentially involves two tasks. We need to find both an ordering of vertices and an assignment of edges to pages that minimizes the total number of pages needed to embed a graph. By allowing modified pages and spines, one may also want to consider the type of book used in addition to vertex-ordering and edge assignment. In the case of the sum of triangles graph D_n, with a certain prespecified vertex-ordering, we see that $bt(D_n)$ can be made arbitrarily large while $ct(D_n) = bt(D_n, P) = 2$ (where P is the tree with two leaves) and $t(G) = 1$. Similarly, the pinwheel graph PW_n has arbitrarily

Figure 4.16 *One-page Klein bottle book embedding of PW_6.*

large book thickness with the vertices lined up on the spine according to the outer circuit. However, with this vertex-ordering, we can embed PW_n on a single page if we select the right type of page.

In this section, we focused on modifications of the pages of books. By selecting the appropriate book, in many cases we were able to greatly reduce the number of pages needed to embed particular graphs. The final chapter of this dissertation explores the problem of finding optimal embeddings for the Cartesian product of two graphs. We examine some bounds on the book-thickness of the Cartesian product of certain bipartite graphs. We show that for graphs with odd cycles, the torus page can be useful in reducing bounds on number of pages needed.

Chapter 5

DISPERSABILITY AND THE CARTESIAN PRODUCT

We have looked at generalizations of the book embedding problem by modifying the spine and the pages. Now we will focus on book embeddings of Cartesian products of graphs. First, we discuss the embedding of such graphs in the standard book. In this discussion, we use techniques of Bernhart and Kainen [2] to give bounds for the standard book thickness of $G \times H$ when one of the two graphs is a dispersable bipartite graph. It is unknown whether all bipartite graphs are dispersable. We present dispersable embeddings of several bipartite graphs and provide some insight into a solution to this problem. Finally, we show how modified pages can be helpful in embedding the Cartesian product of two graphs when both graphs contain odd cycles.

Recall that a graph G with maximal vertex degree k is dispersable if G has a proper k-edge coloring and a k-page book embedding so that all edges of one color lie on the same page. Suppose that G and H are graphs with $bt(G) = m$ and $bt(H) = n$. Can we make any conclusions about $bt(G \times H)$? In general this does not appear to be an easy question. If one of the graphs is a dispersable bipartite graph, Bernhart and Kainen [2] provide the following bound.

Theorem 5.1 *Let H be a dispersable bipartite graph and let G be an arbitrary graph. Then $bt(G \times H) \leq bt(G) + \Delta(H)$, where $\Delta(H)$ is the maximum vertex degree in H.*

Proof: Suppose that H is a dispersable bipartite graph. Since H is dispersable, there is a $\Delta(H)$-edge coloring of H and a corresponding book embedding of H in a $\Delta(H)$-page book so that all edges of one color lie on the same page. Since H is bipartite, there is also a 2-coloring of the vertices of H using the colors black and white.

Now we embed $G \times H$ in the following way. Take a book embedding of G in $bt(G)$ pages. Using a dispersable book embedding of H, replace each white vertex of H with a copy of this book embedding of G and replace each black vertex of H with the same book embedding of G, but in reverse order (i.e. the reflection of the book embedding of G). Now we have a copy of G for each vertex of H. Since each of these copies are placed separately along the spine, the edges of $G \times H$ corresponding to the copies of G can all be fit on $bt(G)$ pages.

The rest of the edges of $G \times H$ connect corresponding vertices in adjacent (with respect to H) copies of G. Since H is bipartite, then edges of H join vertices of different colors. Since the order of the vertices of G is reversed for the two vertex colors in H, the set of edges between two adjacent copies of G corresponding to a single edge of H can all be placed without crossing on a single page as shown in Figure 5.1.

Figure 5.1 *Dispersable four-page book embedding of $K_4 \times C_4$.*

Since the edges of H can be placed on $\Delta(H)$ pages in a dispersable book embedding, then the sets of copies of the edges of H can also be placed on $\Delta(H)$

pages. Hence, all edges of $G \times H$ can be accommodated on $bt(G) + \Delta(H)$ pages. ■

The bound provided in Theorem 5.1 is optimal in some cases. For example, the cube Q_3 has $bt(Q_3) = 2$. But $Q_3 = C_4 \times P$, where C_4 is the circuit of length four and P is the path consisting of a single edge. We see that $bt(C_4) = 1$ and P is dispersable with $\Delta(P) = 1$, so in this case we reach the bound of Theorem 5.1. There are also examples where the actual book thickness of the given Cartesian product of two graphs is less than that given by the theorem. For example, the triangle K_3 has $bt(K_3) = 1$ and the path of length two P_2 is a dispersable bipartite graph with $\Delta(P_2) = 2$. Hence, Theorem 5.1 guarantees a three-page embedding of $K_3 \times P_2$. However, since it is a planar Hamiltonian graph, two pages are sufficient to embed $K_3 \times P_2$.

For the methods of Theorem 5.1 to work we need dispersable bipartite graphs. It is unknown whether all bipartite graphs are dispersable, but several classes of bipartite graphs are known to be. Bernhart and Kainen [2] mention (i) the complete bipartite graph $K_{n,n}$ ($n \geq 1$), (ii) the even circuit C_{2n} ($n \geq 2$), (iii) the binary n-cube $Q(n)$ ($n \geq 0$), and (iv) trees. Since they do not prove that these graphs are dispersable, we provide the details here.

Theorem 5.2 *If $n \geq 1$, then $K_{n,n}$ is dispersable.*

Let $n \geq 1$ be given. The complete bipartite graph $K_{n,n}$ has n vertices of degree n, so $\Delta(K_{n,n}) = n$. Label the $2n$ vertices of $K_{n,n}$ as follows. The n white vertices are labeled $v_1, v_2, \ldots v_n$ and the n black vertices are labeled $u_1, u_2, \ldots u_n$. The vertices will be placed on the spine so that the us and vs alternate, with the ordering $v_1, u_n, v_2, u_{n-1}, \ldots, v_k, u_{n-k+1}, \ldots, v_n, u_1$. Now we will show how to assign the edges to n pages to give a dispersable embedding of $K_{n,n}$.

To determine the assignment of edges, put the vertices in a circle in the order $v_1, u_1, v_2, u_2, \ldots, v_n, u_n$. With respect to the clockwise ordering around the circle, edges joining vertices that are distance one from some v_i can all be placed without crossing on one page of the book embedding. Those edges joining vertices at distance three from a v_i can be placed on another page without crossing. Continue placing edges joining vertices at a particular odd distance from a v_i on a single page. Then we have n non-crossing edges on each page for the n distances $k = 1, 3, \ldots 2n - 1$ as shown in Figure 5.2.

Figure 5.2 *Dispersable four-page book embedding of $K_{4,4}$.*

Thus, all of the n^2 edges of $K_{n,n}$ are placed on n pages without edge crossings so that each vertex has degree at most one on a page. Hence, $K_{n,n}$ is dispersable.

∎

The dispersability of the even circuit C_{2n} can be seen by placing the vertices of the circuit on the spine according to the ordering of the circuit. The edges of C_{2n} are placed on two pages by assigning edges to alternating pages as we follow along the circuit. Since $\Delta(C_{2n}) = 2$, this gives us a dispersable embedding of C_{2n}. Hence the following theorem holds.

Theorem 5.3 *If $n \geq 2$, then C_{2n} is dispersable.* ∎

The binary n-cube Q_n has maximal vertex degree $\Delta(Q_n) = n$. The dispersability of Q_n follows by induction on n.

Theorem 5.4 *If $n \geq 0$, then Q_n is dispersable.*

Proof: If $n = 0$, there is nothing to show. If $n = 1$, then $\Delta(Q_n) = 1$ and one page will clearly work for this graph consisting of two vertices and a single edge. Hence, if $n = 0$ or 1, Q_n is dispersable. For $n \geq 2$ we make the observation that $Q_n = Q_{n-1} \times Q_1$.

Suppose the result holds for $n = 0, 1, \ldots k$. Now consider the graph $Q_{k+1} = Q_k \times Q_1$. By induction, there is a dispersable book embedding of Q_k in a k-page book so that every vertex has degree at most one on each page. We line the vertices of Q_k up on the spine according to such a dispersable embedding. Next, we make a second copy of Q_k, reversing the order of the first dispersable embedding as in Theorem 5.1. We draw edges between corresponding vertices of the copies of Q_k on a single additional page as shown in Figure 5.3.

Now we have an embedding of Q_{k+1} on $k+1$ pages so that no two edges on a page cross and every vertex has degree at most one on a single page. Hence Q_n is dispersable. ∎

Figure 5.3 *Dispersable four-page book embedding of Q_4.*

We know that the square Q_2 is embeddable on one page. Similarly, since Q_3 is a planar graph with a Hamiltonian circuit, it follows that $bt(Q_3) \leq 2$. But Q_3 is not outerplanar, so $bt(Q_3) = 2$. Using the same techniques as in Theorem 5.1 and 5.4 it is easy to see the following bound on the book thickness of Q_n given by Bernhart and Kainen [2].

Theorem 5.5 *If $n \geq 2$, then $bt(Q_n) \leq n - 1$.* ■

Now we show that trees are dispersable by induction on the number of edges in the tree.

Theorem 5.6 *If T is a tree, then T is dispersable.*

Proof: Let T be a tree with n edges. If $n = 0$, there is nothing to show. If $n = 1$, then color the edge and place it on a single page for a dispersable embedding of T.

Now suppose the theorem holds for all trees with $n = 0, 1, \ldots k$ edges. Suppose that T is a tree with $k+1$ edges and suppose that $\Delta(T) = m$. Let v be an endvertex of T and let $T - v$ be the graph formed by removing v and its corresponding edge e from T. Now $T - v$ is a tree with fewer edges than T and $\Delta(T - v)$ is either equal to m or to $m - 1$. Either way, there is a dispersable embedding of $T - v$ in $\Delta(T - v)$ pages. Consider such an embedding of $T - v$.

If u is the vertex of T that is adjacent to v, place v next to u on the spine. Since u must have a degree smaller than m in $T - v$, then there is at least one color in a set of m colors that is not used for any edge incident with u. Use one of these colors for the edge $e = \{u, v\}$ and assign e to the page corresponding to that color. Because v is placed next to u on the spine, the edge e will not cross any other edge on this page. Since no other edge incident with u has the same color as e, the degree of u is at most one on this page. Hence, we have a dispersable embedding of T in m pages. ■

There are other dispersable bipartite graphs beyond those mentioned above. Suppose that $m \leq n$. The graph $K_{m,n}$ has $\Delta(K_{m,n}) = n$. Since $K_{m,n}$ is a subgraph of the graph $K_{n,n}$, we can use an n-page dispersable book embedding of $K_{n,n}$ and delete $n - m$ vertices of one vertex set to give a dispersable book embedding of $K_{m,n}$. Furthermore, if G is any subgraph of $K_{n,n}$ with $\Delta(G) = n$, we can remove the extra vertices and edges from a dispersable book embedding of $K_{n,n}$ to obtain a dispersable book embedding of G.

The definition of dispersability does not specify that the graph must be bipartite. In fact, there are many examples of dispersable graphs that are not bipartite. One such graph is the circuit of length four, C_4, with one diagonal. This graph has maximum vertex-degree three. A dispersable three-page book embedding is easily obtained by two-coloring the circuit and assigning a third color to the diagonal. However, since there are graphs that require $\Delta(G) + 1$ colors for a proper edge coloring, there are graphs that are not dispersable. This set of graphs includes odd cycles and K_n for odd values of n.

If we search further, we see that there are graphs that have proper edge colorings with $\Delta(G)$ colors but do not have dispersable $\Delta(G)$-page book embeddings.

One such graph is K_4. This graph has six edges and maximum vertex degree three. In a dispersable book embedding of K_4, there must be exactly two edges on each of the three pages. Since all four vertices of K_4 are equivalent, we may assume that the vertices are labeled v_1, v_2, v_3, and v_4 and that they appear in this order along the spine. The edge $e = \{v_1, v_3\}$ must appear on one of the pages. The edge e now blocks v_2, preventing the possibility of two edges on this page in a dispersable embedding of K_4. Hence, K_4 is not dispersable.

Similarly, K_{2m} $(m \geq 2)$ is not dispersable. It is interesting to note that for both odd and even values of n, only one more page is needed to obtain a book embedding of K_n so that every vertex has degree at most one on each page. This can be done by lining the vertices up on the spine in the order $v_1, v_2, \ldots, v_n$. We embed K_n on n pages in the following way. For $k = 0, 1, \ldots, n-1$, draw an edge between v_i and v_j on page k if $i + j \equiv k \pmod{n}$. The edges on a single page do not cross as shown in Figure 5.4.

When considering a regular graph G, we make the observation that G is dispersable only if G is bipartite. This follows from the fact that in a dispersable embedding of a regular graph, every vertex must have degree one on every page. Hence, two vertices that are joined by an edge must be at odd distances from each other along the spine. If G has an odd cycle, then any two adjacent vertices can be placed at odd distances from each other along the spine, until the last vertex, which is an even distance from the first vertex. Then the edge joining these two vertices will block off an odd number of vertices on that page, preventing the maximal placement of edges. Hence, G cannot have an odd cycle, and a regular dispersable graph must be bipartite.

With this observation and since there are numerous examples of non-bipartite graphs that are not dispersable, we return to our study of bipartite graphs. Every

Figure 5.4 *Nearly dispersable book embeddings of K_n in n pages.*

bipartite graph with maximal vertex degree k has a proper k-edge coloring. It is not clear that these graphs have dispersable k-page book embeddings. Since there are no known examples of non-dispersable bipartite graphs, we attempted to either prove that all bipartite graphs are dispersable or find a counter-example. We have examples of edge colorings of $K_{4,4}$ with four colors that do not yield a dispersable four-page book embedding with any vertex-ordering along the spine. Hence, not every k-coloring of a bipartite graph G with $\Delta(G) = k$ yields a dispersable k-page book embedding. By Theorem 5.2, there is a dispersable embedding of $K_{4,4}$ in a 4-book. Although this graph is not a counter-example, it does show that we cannot prove the result by taking an arbitrary k-edge coloring of a graph G with $\Delta(G) = k$ and find a corresponding dispersable k-page book embedding of G.

We have already shown that regular dispersable graphs are bipartite. But, is the converse true? Are all regular bipartite graphs dispersable? Since there are many possible edge-colorings of bipartite graphs and many possible vertex-orderings, it is difficult to determine whether a particular graph is actually not dispersable. We focus our attention on regular bipartite graphs because all bipartite graphs are dispersable if all regular bipartite graphs are. This is true since every bipartite graph with maximal vertex degree k is a subgraph of a regular bipartite graph with maximal degree k.

We note that any k-regular subgraph of $K_{n,n}$ has $2n$ vertices of degree k. Hence, any dispersable k-page book embedding of such a graph must have n edges on each of the k pages so that every vertex has degree one on every page of the k-page embedding. It seems likely that for some regular subgraph of $K_{n,n}$ this is impossible. In an attempt to find one, we conducted a computer assisted search which revealed that every regular subgraph of $K_{n,n}$ ($n \leq 6$) is dispersable. Beyond $n = 6$, we are uncertain whether arbitrary k-regular subgraphs are dispersable. However, for any n, there are certain values of k for which k-regular subgraphs of $K_{n,n}$ are dispersable.

All one-regular subgraphs of $K_{n,n}$ are dispersable since these graphs consist only of n disjoint edges that may be placed without crossing on a single page. Similarly, two-regular subgraphs of $K_{n,n}$ have dispersable two-page book embeddings. This is clear since such graphs consist of disjoint even cycles which each have dispersable two-page book embeddings. Since $K_{n,n}$ is dispersable, then the only n-regular subgraph of $K_{n,n}$ is dispersable. There is only one possible $n-1$-regular subgraph of $K_{n,n}$ up to isomorphism. Since this is the subgraph formed by removing any page of the dispersable n-page book embedding of $K_{n,n}$ of Theorem 5.2, it has the corresponding $(n-1)$-page dispersable book embedding. From here it is not clear how to proceed to find dispersable embeddings for k-regular subgraphs of $K_{n,n}$ when $3 \leq k \leq n-2$, so the question remains open.

The techniques of Theorem 5.1 are useful for finding book embeddings of the Cartesian product of two graphs when one of the graphs is a dispersable bipartite graph. What about the case when both graphs contain an odd cycle? In Theorem 5.1, to embed $G \times H$ in a book, we use the fact that H is bipartite to reverse the vertex-orderings of adjacent copies of G. If H has an odd cycle, then the vertices of H cannot be two-colored. Thus, the technique of reversing the vertex-ordering of adjacent copies of G will result in two adjacent copies of G with the same vertex-ordering. Each edge between two corresponding vertices in adjacent copies of G with the same vertex-ordering must lie on its own page. Thus, if G has many vertices, the number of pages required with this embedding scheme will grow as the number of vertices of G grows. If we change the type of page, we can handle odd cycles more efficiently.

Consider the Cartesian product of a graph G with book thickness $bt(G) = k$ and an odd cycle C_{2m+1}. The odd cycle has $\Delta(C_{2m+1}) = 2$, but three colors are required to color the edges of C_{2m+1}. Similarly, three colors are needed to color the vertices of C_{2m+1}. So, C_{2m+1} is neither dispersable nor bipartite. If we alternate the ordering of the vertices of G along the circuit C_{2m+1} as in Theorem 5.1, we will need at least $\|V(G)\|$ pages for the edges between the two adjacent copies of G that have the same vertex-ordering. The next theorem shows how a book with torus pages can reduce the number of pages needed.

Theorem 5.7 *Let G be a graph with $bt(G) = k$ and let $m \geq 1$. Then $t(G \times C_{2m+1}) \leq \lceil k/2 \rceil + 1$.*

Proof: Since G has a standard book embedding in k pages, then G is embeddable in a $\lceil k/2 \rceil$-page torus book using this k-page embedding. Hence, $t(G) \leq \lceil k/2 \rceil$.

Now place the vertices of C_{2m+1} along a torus spine. Replace each vertex of C_{2m+1} with a copy of G, where the vertices of G are lined up corresponding to a $t(G)$-page torus book embedding of G. Each copy of G is placed with the same vertex-ordering. The edges joining corresponding vertices of G can all be placed on a single torus page without crossing as shown in Figure 5.5.

Figure 5.5 *Two-page torus book embedding of $K_4 \times C_5$.*

Now we see that $t(G \times C_{2m+1}) \leq t(G) + 1 \leq \lceil k/2 \rceil + 1$. ∎

Theorem 5.7 could have been stated in terms of torus pages in the following way.

Theorem 5.8 *Let G be a graph with $t(G) = k$ and let $m \geq 1$. Then $t(G \times C_{2m+1}) \leq k + 1$.* ∎

Instead of one additional torus page for the edges between the copies of G, we could have used two cylinder pages. For graphs G and H with multiple odd circuits, general bounds for $t(G \times H)$ and $ct(G \times H)$ are not as easy to obtain. However, both the torus book and the cylinder book appear to be better for handling such graphs than the standard book. Again, as we have seen in the previous chapter, minimizing the total number of pages needed to embed a certain graph can often be simplified by using books with modified pages.

BIBLIOGRAPHY

[1] M. Aigner, *Graph Theory: A Development from the 4-Color Problem*, BCS Associates, Moscow, Idaho (1987).

[2] F. Bernhart and P. C. Kainen, *The Book Thickness of a Graph*, Journal of Combinatorial Theory, Series B **27** (1979) no. 3, 320–331.

[3] J. F. Buss, A. L. Rosenberg, and J. D. Knott, *Vertex-Types in Book Embeddings*, SIAM Journal on Discrete Mathematics **2** (1989) no. 2, 156–175.

[4] J. F. Buss and P. W. Shor, *On the Pagenumber of Planar Graphs*, Proceedings of the 16th Annual ACM Symposium on Theory of Computing (Washington, D.C., 1984), ACM , New York, 1984, 98–100.

[5] F. R. K. Chung, F. T. Leighton, and A. L. Rosenberg, *Diogenes: A Methodology for Designing Fault-Tolerant VLSI Processor Arrays*, Proceedings of the 13th International Conference on Fault-Tolerant Computing (Milan, Italy, 1983), IEEE, New York, 1983, 26–32.

[6] F. R. K. Chung, F. T. Leighton, and A. L. Rosenberg, *Embedding Graphs in Books: A Layout Problem with Applications to VLSI Design*, SIAM Journal on Algebraic and Discrete Methods **8** (1987) no. 1, 33–58.

[7] R. A. Games, *Optimal Book Embeddings of the FFT, Benes, and Barrel Shifter Networks*, Algorithmica **1** (1986), 233–250.

[8] M. R. Garey, D. S. Johnson, G. L. Miller, and C. H. Papadimitriou, *The Complexity of Coloring Circular Arcs and Chords*, SIAM Journal on Algebraic and Discrete Methods **1** (1980) no. 2, 216–227.

[9] F. Harary and G. Prins, *The Number of Homeomorphically Irreducible Trees, and Other Species*, Acta Mathematica, **101** (1959), 161.

[10] L. S. Heath, *Embedding Outerplanar Graphs in Small Books*, SIAM Journal on Algebraic and Discrete Methods **8** (1987) no. 2, 198–218.

[11] L. S. Heath, *Embedding Planar Graphs in Seven Pages*, Proceedings of the 25th Annual IEEE Symposium of Foundations of Computer Science (Singer Island, Florida, 1984), IEEE, New York, 1984, 74–83.

[**12**] L. S. Heath and S. Istrail, *The Pagenumber of Genus g Graphs is $O(g)$*, Journal of the Association for Computing Machinery **39** (1992) no. 3, 479–501.

[**13**] D. A. Holton, B. Manvel, and B. D. McKay, *Hamiltonian Cycles in Cubic* 3-*Connected Bipartite Planar Graphs*, Journal of Combinatorial Theory, Series B **38** (1985), no. 3, 279-297.

[**14**] L. T. Q. Hung, *A Planar Poset Which Requires Four Pages*, Ars Combinatoria **35** (1993), 291–302.

[**15**] P. C. Kainen, personal communication.

[**16**] P. C. Kainen, *The Book Thickness of a Graph II*, Proceedings of the 20th Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, Florida, 1989), Congressus Numerantium **71** (1990), 127–132.

[**17**] R. Kannan, *Unraveling k-Page Graphs*, Information and Control **66** (1985), 1–5.

[**18**] K. S. Kedlaya, *Outerplanar Partitions of Planar Graphs*, Journal of Combinatorial Theory, Series B **67** (1996) no. 2, 238-248.

[**19**] S. M. Malitz, *Genus g Graphs Have Pagenumber $O(\sqrt{g})$*, Proceedings of the 29th Annual IEEE Symposium on Foundations of Computer Science (White Plains, New York, 1988), IEEE, New York, 1988, 458–468.

[**20**] S. Moran and Y. Wolfsthal, *One-Page Book Embeddings Under Vertex-Neighborhood Constraints*, SIAM Journal on Discrete Mathematics **3** (1990) no. 3, 376–390.

[**21**] S. Moran and Y. Wolfsthal, *Two-Page Book Embeddings of Trees Under Vertex-Neighborhood Constraints*, Discrete Applied Mathematics **43** (1993) no. 3, 233–241.

[**22**] D. J. Muder, M. L. Weaver, and D. B. West, *Pagenumber of Complete Bipartite Graphs*, Journal of Graph Theory **12** (1988) no. 4, 469-489.

[**23**] B. Obrenic, *Embedding DeBruijn and Shuffle-Exchange Graphs in Five Pages*, SIAM Journal on Discrete Mathematics **6** (1993) no. 4, 642–654.

[**24**] L. T. Ollmann, *On the Book Thickness of Various Graphs*, Proceedings of the 4th Southeastern Conference on Combinatorics, Graph Theory, and Computing, Utilitas Mathematica Publishing, Inc., Winnipeg, 1973, 459.

[**25**] A. L. Rosenberg, *Book Embeddings and Wafer-Scale Integration*, Proceedings of the 17th Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, Florida, 1986), Congressus Numerantium **54** 1986, 217–224 .

[**26**] A. L. Rosenberg, *The Diogenes Approach to Testable Fault-Tolerant Arrays of Processors*, IEEE Transactions on Computers **C-32** (1983) no. 10, 902–910.

[**27**] E. Stöhr, *A Trade-off between Page Number and Page Width of Book Embeddings of Graphs*, Information and Computation **79** (1988), 155–162.

[**28**] M. M. Syslo, *Characterizations of Outerplanar Graphs*, Discrete Mathematics **26** (1979), 47–53.

[**29**] W. T. Tutte, *A Theorem on Planar Graphs*, Transactions of the American Mathematical Society **82** (1956), 99–116.

[**30**] J. H. Van Lint and R. M. Wilson, *A Course in Combinatorics*, Cambridge University Press (1992).

[**31**] H. Whitney, *A Theorem on Graphs*, Annals of Mathematics **32** (1931), 378–390.

[**32**] M. Yannakakis, *Four Pages are Necessary and Sufficient for Planar Graphs*, Proceedings of the 18th Annual ACM Symposium on Theory of Computing (Berkley, California, 1986), ACM, New York, 1986, 104–108.

[**33**] M. Yannakakis, *Linear and Book Embeddings of Graphs*, VLSI Algorithms and Architectures (Loutraki, 1986), 226–235, *Lecture Notes in Computer Science, Vol. 227*, Springer, Berlin, 1986.