False. If S is a linearly dependent set, then some vector of S is a linear combination of the other vectors of S.

True, See pg 59.

True. See p. 43.

True, See p. 58.

True. See p. 65.

True, See p. 46.

True, See p. 66.

True. See p. 66.

False. Here is one counterexample: \( A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \vec{x} = \begin{pmatrix} 0 \\ 17 \end{pmatrix}, A\vec{x} = \begin{pmatrix} 0 \end{pmatrix} \), but \( \vec{x} \) as zero as one of its entries.

True. The solution \( x_1 = x_2 = x_4 = 0, x_3 = 17 \) is a nontrivial solution to \( x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 = \vec{0} \) and so \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) is linearly dependent since a non-trivial solution exists.

False. Let \( \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \). Then \( \vec{v}_3 \neq a \vec{v}_1 + a \vec{v}_2 + a \vec{v}_3 \) for any \( a, a, a \), real numbers, but the set \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) is linearly dependent since \( 2 \vec{v}_1 - \vec{v}_2 + \vec{v}_3 = 1 \) is a nontrivial linear dependence relation.

True! \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) are linearly independent vectors, then the only solution to \( a \vec{v}_1 + b \vec{v}_2 + c \vec{v}_3 = \vec{0} \) is the trivial solution (because the only solution to \( a \vec{v}_1 + b \vec{v}_2 + c \vec{v}_3 = \vec{0} \) is the trivial one) & so \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) is also linearly independent.

First, we must solve the system \( \begin{pmatrix} 1 & 3 & -3 & 0 \\ -3 & -1 & 9 & 0 \end{pmatrix} \) reduces to \( \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \end{pmatrix} \). Thus, the solution is \( x_1 = -4x_3, x_2 = 3x_3, \) & \( x_3 \) is free in parametric form, this is:

\[ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix} \]

\( \begin{pmatrix} 1 & -2 & 9 & 0 \\ 0 & 1 & 2 & -6 \\ 0 & 1 & 6 & 0 \end{pmatrix} \) reduces to \( \begin{pmatrix} 1 & 0 & -5 & -7 \\ 0 & 1 & 2 & -6 \\ 0 & 2 & 0 & -3 \end{pmatrix} \) so the parametric solution is \( \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \\ -3 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \).

The line in parametric vector form is \( \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \).

The line is \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} + t \begin{pmatrix} 0 \\ 8 \end{pmatrix} \).

A vector parallel to the line is \( \begin{pmatrix} 1 \\ 3 \end{pmatrix} \), \( \begin{pmatrix} 0 \\ -4 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \end{pmatrix} \), so the line can be written as \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} + t \begin{pmatrix} -6 \\ -3 \end{pmatrix} \).

Let \( \vec{p} \) be a solution to \( A \vec{x} = \vec{b} \). A theorem says the set of all solutions to \( A \vec{x} = \vec{b} \) is the set of vectors of the form \( \vec{p} + \vec{v} \vec{f} \) where \( \vec{f} \) is a solution to \( A \vec{x} = \vec{0} \). So the number of solutions to \( A \vec{x} = \vec{b} \) and \( A \vec{x} = \vec{0} \) are the same. In particular, \( A \vec{x} = \vec{b} \) has only one solution exactly when \( A \vec{x} = \vec{0} \) has only one solution (the trivial one).

All vectors in \( \mathbb{R}^3 \) are in the solution set since \( \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \) for all \( \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \).

\( \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} \) is included in \( \text{Span} \left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} \right\} \) if and only if the system with augmented matrix \( \begin{pmatrix} 3 & -3 & 5 \\ -2 & -6 & 0 \end{pmatrix} \) is consistent. The first row says \( x_1 - 3x_2 = 5 \). Multiplying this by -3 yields \( -3x_1 + 9x_2 = -15 \), but the second row says \( -3x_1 + 9x_2 = -7 \). Thus, the system is inconsistent for any \( a \) & the vector is never in the span. The set is linearly dependent for all values of \( a \) since the
The set is linearly dependent when the system with augmented matrix \[
\begin{bmatrix}
2 & -6 & 8 & 0 \\
1 & 7 & 3 & 4 \\
0 & 0 & 2 & 0
\end{bmatrix}
\] has a non-trivial solution. This matrix reduces to \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\] so the system has a free variable (and hence non-trivial solution) when \(16 + a = 0\). That is, the set is linearly dependent when \(a = -16\).

The columns of a \(5 \times 7\) matrix are never linearly independent because there are more vectors than entries in each vector.

Let \(A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\), \(B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\) then \(Tv = \begin{bmatrix} 2 \\ -2 \end{bmatrix}\) is a non-trivial solution to \(AV = 0\) but writing \(x = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}\), we know \(Bx = \overrightarrow{0}\) says \(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\) or equivalently \(x_1 = 0 \& x_2 = 0\), so \(Bx = \overrightarrow{0}\) has only the trivial solution.

In particular, the hypothesis says \(Ax = \overrightarrow{0}\) has at most one solution. Writing this as a vector equation in the columns of \(A = \begin{bmatrix} a_1 & \ldots & a_n \end{bmatrix}\) we have \(x_1a_1 + \cdots + x_na_n = \overrightarrow{0}\) has at most one (and hence exactly one since the system is homogeneous) solution. Thus, the columns of \(A\) are linearly independent by definition.

We need to find all solutions to the system with augmented matrix \[
\begin{bmatrix}
1 & 3 & 9 & 2 & 8 & 0 \\
0 & 1 & 2 & 3 & 4 & 5 \\
2 & 3 & 6 & 0 & 0 & 0
\end{bmatrix}
\] which reduces to \[
\begin{bmatrix}
1 & 0 & 9 & 2 & 8 & 0 \\
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\] so the solutions are of the form: \(x_1 = -3x_3, \ x_2 = 2x_3, \ x_3 \text{ free}, \ x_4 = 0\)

The vector is in the range if the system with augmented matrix \[
\begin{bmatrix}
1 & 3 & 9 & 2 & 8 & 0 \\
0 & 1 & 2 & 3 & 4 & 5 \\
2 & 3 & 6 & 0 & 0 & 0
\end{bmatrix}
\] is consistent. This matrix reduces to \[
\begin{bmatrix}
1 & 0 & 9 & 2 & 8 & 0 \\
0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\] so \(v\) is not in the range of the linear transformation \[
\begin{bmatrix}
2 & 3 & 0 & 0 & 5 & 0
\end{bmatrix}
\]

Let \(A = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}\) then \(A \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2x_1 + 7x_2 \\ -5x_1 + 3x_2 \end{bmatrix} = x_1 \begin{bmatrix} -2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ -3 \end{bmatrix} = x_1v_1 + x_2v_2\) as desired.

Let \(x \in \mathbb{R}^n\). As \(v_1, \ldots, v_3\) spans \(\mathbb{R}^3\), then \(x = a_1v_1 + \cdots + a_3v_3\) for some scalars \(a_1, \ldots, a_3\). Also, \(T(x) = a_1Tv_1 + \cdots + a_3Tv_3 = \overrightarrow{0}\).

Re-number \(v_1, v_2, v_3\) so that \(v_3\) is a linear combination of the other two, and write \(v_3 = a_1v_1 + a_2v_2\). Then \(Tv_3 = a_1Tv_1 + a_2Tv_2\) and \(Tv_3 = a_1Tv_1 + a_2Tv_2\) so that \(Tv_3\) is a linear combination of \(Tv_1\) and \(Tv_2\). Thus \(Tv_1, Tv_2, Tv_3\) is linearly dependent.