1. True. See p. 70.
2. False. Let \( T_1: \mathbb{R}^n \rightarrow \mathbb{R}^m \) and \( T_2: \mathbb{R}^m \rightarrow \mathbb{R}^k \) be linear transformations. Then, \( T_2 \circ T_1 \) is a linear transformation because:

\[
T_2(aT_1(x) + bT_1(x)) = aT_2(T_1(x)) + bT_2(T_1(x)) = aT_2(T_1(x)) + bT_2(T_1(x))
\]

showing that a composition of linear transformations is again a linear transformation.

3. False. Let \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) and \( \alpha = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^2 \). Then \( \alpha \mapsto A\alpha = \begin{bmatrix} x_3 \\ 0 \end{bmatrix} \) is zero exactly when \( x_3 = 0 \). Thus, the linear transformation \( x_2 \mapsto A\alpha \) is not injective.

4. False. Let \( A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} \). Then \( AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = AC \), but \( B \neq C \).

5. The standard matrix representation is \( \begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix} \).

6. Since \( T_1(x) = T(\alpha) \) and \( T_2(x) = T(\beta) \), we're looking for a system \( T(-\alpha, 3\beta) = -\alpha + 3\beta \).

7. \( T(\alpha) = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \) when the system with augmented matrix \( \begin{bmatrix} 1 & -2 & -1 \\ 3 & -2 & 4 \end{bmatrix} \) has a solution.

8. This matrix reduces to \( \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) and so \( \alpha = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \) has the property that \( T(\alpha) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \).

9. \( T \) is injective if and only if \( T(\alpha) = \overline{\alpha} \) has only the trivial solution. The matrix \( \begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 0 \end{bmatrix} \) reduces to \( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) so \( x_1 = x_2 = 0 \) is the only solution. \( T \) is surjective if and only if the matrix \( \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 0 \end{bmatrix} \) represents a consistent system for all real numbers \( a, b, c \). With \( a+b=0 \) and \( c=1 \), this reduces to \( \begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & 1 \end{bmatrix} \) which is not the augmented matrix of a consistent system, so \( T \) is not surjective.

10. Here, \( T \) is not injective since \( \alpha = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) is a nonzero vector such that \( T(\alpha) = \overline{\alpha} \). \( T \) is surjective if and only if the matrix \( \begin{bmatrix} 1 & 4 & -5 & a \\ 3 & -7 & 4 & b \end{bmatrix} \) is the augmented matrix of a consistent system for all real numbers \( a, b \). This matrix reduces to \( \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \).

11. \( T \) is one-to-one if and only if \( A \) has \( n \) pivot columns. For the explanation: The matrix \( A \) has \( n \) pivot columns if and only if every column of \( A \) is a pivot column. This in turn is equivalent to the equation \( Ax = \overline{\alpha} \) having no free variables, i.e., having a unique solution, which is exactly what is required for \( T \) to be one-to-one.

12. \( T \) is surjective if and only if \( A \) has \( m \) pivot columns. For the explanation: \( A \) has \( m \) pivot columns if and only if \( A \) has a pivot position in every row. This is in turn equivalent to the columns of \( A \) spanning \( \mathbb{R}^m \) which is equivalent to having \( T \) as a surjective map.
9. In (8), we saw that for \( T \) to be surjective (onto), the standard matrix representation must have \( m \) pivot columns. This can only happen when \( m = n \). Similarly, in (9), we saw that for \( T \) to be injective (one-to-one), the standard matrix representation must have \( n \) pivot columns. This can only happen when \( n = m \).

10. \( A - 2A = \begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix} \)

11. \( B - 2A = \begin{bmatrix} 7 & -5 & 1 \\ 1 & 4 & -3 \end{bmatrix} - \begin{bmatrix} -8 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 3 \\ -7 & 14 & -7 \end{bmatrix} \)

12. \( AC \) is undefined because the number of columns of \( A \) does not equal the number of rows of \( B \).

13. \( CD = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 13 \\ -5 & -6 \end{bmatrix} \)

14. Suppose \( CA = I_n \) & \( x \) is a vector such that \( Ax = 0 \). Then, \( CAx = 0 \). But \( CAx = (CA)x = I_n x = x \). Thus, \( x = 0 \) & the equation \( Ax = 0 \) has only the trivial solution. In the language of linear transformations, \( A \) is the standard matrix representation of an injective linear transformation. If \( A \) is an \( m \times n \) matrix, then problem 14 tells us that \( n \leq m \), i.e., \( A \) cannot have more columns than rows.

15. Suppose \( AD = I_m \) and \( \vec{b} \in \mathbb{R}^m \). Let \( \vec{x} = DB \). Then \( A\vec{x} = A(DB) = (AD)\vec{b} = I_m \vec{b} = \vec{b} \). In the language of linear transformations, \( A \) is the standard matrix representation of a surjective linear transformation. If \( A \) is an \( m \times n \) matrix, then problem 15 tells us that \( m \geq n \), i.e., \( A \) cannot have more rows than columns.

16. The entry in the \( i \)-th row & \( j \)-th column of \( I_m A \) is found by taking the dot product of the \( i \)-th row of \( I_m \) with the \( j \)-th column of \( A \). The \( i \)-th row of \( I_m \) has a 1 in the \( i \)-th entry & 0's elsewhere. Thus, the dot product with the \( j \)-th column of \( A \) is just \( a_{ij} \).

17. Alternate (originally intended) proof: Write \( A \) as \( [\vec{a}_1 \vec{a}_2 \ldots \vec{a}_m] \). Then by definition \( I_m A = [I_m \vec{a}_1 \ I_m \vec{a}_2 \ldots \ I_m \vec{a}_m] \). Since \( I_m \vec{a}_i = \vec{a}_i \) for \( i = 1, \ldots, m \), then \( I_m A = [\vec{a}_1 \ \vec{a}_2 \ldots \ \vec{a}_m] = A \).