Linear Algebra HW#6 Solutions

1. False. If A is a 2x3 matrix and B is a 3x4 matrix, then $A^T$ is a 3x2 matrix & $B^T$ is a 4x3 matrix, so the product $A^TB^T$ is not even defined.


3. False. Let $A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}$, then $A^T = \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$, $B^T = \begin{bmatrix} 5 & -1 \\ 1 & 0 \end{bmatrix}$, $AB = \begin{bmatrix} 7 & 40 \\ -3 & 17 \end{bmatrix}$, $A^TB^T = \begin{bmatrix} 8 & -3 \end{bmatrix}$.

$(AB)(A^TB^T) = \begin{bmatrix} 256 & 101 \\ -109 & 43 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so $A^TB^T$ is not the inverse of $AB$.

4. True. In fact $A = A^T$ is always the unique solution since $A(A^TB) = (AA^T)B = I_n B = B$.


6. True. See (8) of Thm 8 on p. 112.

7. True. See (8) of Thm 8 on p. 112.

8. True. Thm 8 (8) says the linear transformation $x \rightarrow Ax$ is injective. Equivalently, $Ax = B$ has only one solution for each $B$.

9. The hypothesis says (8) of Thm 8 is false. Thus (8) is false as well (i.e., the linear transformation with standard matrix representation $A$ is injective).

$(ABx)^T = x^T B^T A^T$ since we can view $Ax$ as a matrix as one column, then Thm 3 on pg 99 tells us the transpose of a product is the product of the transposes in the opposite order.

10. $7x + 2y = 22, 4x + 3y = 6$, so $\begin{bmatrix} 7 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 22 \\ 6 \end{bmatrix}$. The system $7x + 3y = 22$ has unique solution $\begin{bmatrix} 22 \\ 2 \end{bmatrix}$.

11. $B = I_n B = (A^T A) B = A^T (A B) = A^T (A C) = I_n C = C$. If $A$ is not invertible, this may be false. See HW #5, problem (8).

12. Let $C = AB$. Since $B$ is invertible, $C^T = (AB)^T = B^T A^T = A^T B = A$. As $C = AB$ is invertible, then $A = C B^T$ is the product of two invertible matrices & hence is invertible as well (see Thm 6 p. 105).


14. Since $(A - X) X^T = X^T B$ on the left by $X$ yields $X(A - X) = X(X^T B) = (XX^T) B = I_n B = B$. Since $B$ is the product of two invertible matrices, it must also be invertible. Moreover, $B = (A - X) X^T = (A - X)^T$. To solve for $X$, we multiply both sides of $X(A - X) = B$ by $(A - X)$, to yield $B(A - X) = [X (A - X)](A - X) = X [A(A - X)] - X I_n = X$.

15. When $A$ is invertible, it is row equivalent to $I_n$. Thus, for any $b \in \mathbb{R}^n$, we have $x_1, ..., x_n \in \mathbb{R}$ has a solution (where $\vec{a}_1, ..., \vec{a}_n$ are the columns of $A$). Indeed, the matrix form of this equation is $A \vec{x} = \vec{b}$ which has the unique solution $A^{-1} \vec{b}$.

16. The third column of $A^T$ can be found by solving $A \vec{z} = \vec{e}_3$. The augmented matrix of this equation is $\begin{bmatrix} 2 & 7 & -9 & 0 \\ 1 & 5 & 6 & 8 \\ 0 & 1 & 4 \end{bmatrix}$, which reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$, so the third column of $A^T$ is $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$.

17. $(4 - 6) = \begin{bmatrix} 4 & -6 \\ -1 & 0 \end{bmatrix}$ Not invertible. See Thm 8.

18. Not invertible. The middle column is $0$ so the columns do not form a linearly independent set.
A square lower triangular matrix is invertible when none of the diagonal entries are 0. Each nonzero diagonal entry can be used to clear out the nonzero entries below it. If a diagonal entry is zero, it will not become nonzero in this process, and so we will be left with fewer than n pivot positions.

If the columns of a 5x5 matrix do not span \( \mathbb{R}^5 \), then condition (2) of Thm 8 is not satisfied, so (2) cannot be true either. That is, A will not be invertible.

If A is invertible, then \( A^t \) is also invertible & Thm 8 part (2) says the columns of \( A^t \) thus form a linearly independent set.

If \( EF=I \), then \( F \) is invertible (by Thm 8 (1) => (2)). So \( E \) is the unique inverse of \( F \) and by definition \( FE=I \) as well.

For any matrix \( L \), the equation \( Lx=0 \) has the trivial solution, so \( L \) may or may not have linearly independent columns. If the problem instead said that \( Lx=0 \) had ONLY the trivial solution, then we could conclude by Thm 8 (1) => (2) that the columns of \( L \) span \( \mathbb{R}^n \).

If \( AB \) is invertible, then there is a matrix \( W \) (by definition) such that \( (AB)W=W(AB)=I \). Rewriting this as \( A(BW)=I \) and \( (WA)B=I \), we see from Thm 8 (2) => (2) & (2) => (2) (respectively) that \( A \) & \( B \) (resp) are invertible.

Let \( A \) be the given matrix To find \( A^t \) in Sagemath, either use the code \( A.inverse() \) or define the 5x10 matrix with \( A \) in the first 5 columns & \( I_5 \) in the remaining columns and row reduce that matrix. The result of either computation should tell you that \( A^t=\begin{bmatrix} -662 & -\frac{765}{8} & 19 & 1 & 683 \\ 141 & 815 & -41 & -2 & -1455 \\ -700 & -404 & 21 & 1 & 721 \\ -33 & -19 & 1 & 0 & 34 \\ 1 & \frac{1}{2} & 0 & 0 & -1 \end{bmatrix} \)