1. True or False. Justify each answer with an explanation or counterexample (as appropriate).

(a) A subspace is also a vector space.
(b) A vector is any element of a vector space.
(c) \( \mathbb{R}^2 \) is a subspace of \( \mathbb{R}^3 \).
(d) The null space of an \( m \times n \) matrix \( A \) is the solution set of the equation \( A\vec{x} = \vec{0} \).
(e) The kernel of a linear transformation is a vector space.
(f) \( \text{Col}(A) \) is the set of all solutions of \( A\vec{x} = \vec{b} \).

2. Let \( H \) be the set of all vectors of the form
\[
\begin{pmatrix}
-2t \\
5t \\
3t \\
\end{pmatrix}
\]
Find a vector \( \vec{v} \in \mathbb{R}^3 \) such that \( H = \text{Span}\{\vec{v}\} \). Why does this show that \( H \) is a subspace of \( \mathbb{R}^3 \)?

3. Is the set of all polynomials of the form \( p(t) = at^2 \) where \( a \in \mathbb{R} \) a subspace of \( \mathbb{P}_n \) for some value of \( n \)? Explain your answer.

4. Is the set of all polynomials of the form \( p(t) = a + t^2 \) where \( a \in \mathbb{R} \) a subspace of \( \mathbb{P}_n \) for some value of \( n \)? Explain your answer.

5. For fixed positive integers \( m \) and \( n \), the set \( M_{m\times n} \) of all \( m \times n \) matrices is a vector space, under the usual operations of addition of matrices and multiplication by real scalars. Determine if the set \( H \) of all matrices of the form
\[
\begin{pmatrix}
a & b \\
0 & d \\
\end{pmatrix}
\]
is a subspace of \( M_{2\times 2} \).

6. Define \( T : M_{2\times 2} \to M_{2\times 2} \) by \( T(A) = A + A^T \). Note: An arbitrary element of \( M_{2\times 2} \) is of the form
\[
\begin{pmatrix}
a & b \\
c & d \\
\end{pmatrix}
\]

(a) Show that \( T \) is a linear transformation.
(b) Let \( B \) be any element of \( M_{2\times 2} \) such that \( B^T = B \). Find \( A \) in \( M_{2\times 2} \) such that \( T(A) = B \).
(c) Show that the range of \( T \) is the set of \( B \) in \( M_{2\times 2} \) with the property that \( B^T = B \).
(d) Describe the kernel of \( T \).

7. Let \( \vec{u} \) be an element in a vector space \( V \). Suppose \( c\vec{u} = \vec{0} \) for some nonzero scalar \( c \). Show that \( \vec{u} = \vec{0} \). Mention the axioms or properties you use.

8. Determine if \( \vec{w} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \) is in \( \text{Nul}(A) \) where
\[
A = \begin{pmatrix}
2 & 6 & 4 \\
-3 & 2 & 5 \\
-5 & -4 & 1 \\
\end{pmatrix}
\]

9. Find an explicit description of \( \text{Nul}(A) \) where \( A = \begin{pmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -1 \end{pmatrix} \) by listing vectors that span the null space.
10. Use either an appropriate theorem to show that the given set is a vector space, or find a specific example to the contrary.

(a) \( H = \left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : r, s, t \in \mathbb{R}, 3r - 2 = 3s + t \right\} \)

(b) \( K = \left\{ \begin{bmatrix} c - 6d \\ d \\ c \end{bmatrix} : c, d \in \mathbb{R} \right\} \)

11. Let \( A = \begin{bmatrix} 5 & -2 & 3 \\ -1 & 0 & -1 \\ 0 & -2 & -2 \\ -5 & 7 & 2 \end{bmatrix} \). What are the values of \( k \) and \( t \) such that \( \text{Nul}(A) \) is a subspace of \( \mathbb{R}^k \) and \( \text{Col}(A) \) is a subspace of \( \mathbb{R}^t \).

12. Define \( T : \mathbb{P}_2 \rightarrow \mathbb{R}^2 \) by \( T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} \). For instance, if \( p(t) = 3 + 6t + 8t^2 \), then \( T(p) = \begin{bmatrix} 3 \\ 17 \end{bmatrix} \).

(a) Show that \( T \) is a linear transformation. [Hint: For arbitrary polynomials \( p, q \in \mathbb{P}_2 \), compute \( T(p + q) \) and \( T(cp) \).]

(b) Find a polynomial \( p \in \mathbb{P}_2 \) that spans the kernel of \( T \) and describe the range of \( T \).

13. Let \( T : V \rightarrow W \) be a linear transformation from a vector space \( V \) into a vector space \( W \). Prove that the range of \( T \) is a subspace of \( W \). [Hint: Typical elements of the range have the form \( T(v) \) and \( T(w) \) for some \( v, w \in V \).]