HOMOLOGICAL CHARACTERIZATIONS OF
QUASI-COMPLETE INTERSECTIONS

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Abstract. Let $R$ be a commutative ring, $(f)$ an ideal of $R$, and $E = K(f; R)$ the Koszul complex.
We investigate the structure of the Tate construction $T$ associated with $E$. In particular, we study
the relationship between the homology of $T$, the quasi-complete intersection property of ideals, and
the complete intersection property of (local) rings.

Introduction

Let $(R, m)$ be a commutative, Noetherian, local ring with maximal ideal $m$ and let $I$ be a proper,
non-zero ideal of $R$. Fix a generating set $f$ of $I$, and let $E$ be the Koszul complex on $f$.
Recall that $I$ is a complete intersection ideal if it can be generated by a regular sequence. As $R$
is local, this condition is tantamount to the following (equivalent) conditions:

1. $H_i(E) = 0$ for all $i > 0$.
2. $H_1(E) = 0$.

Let $S$ denote $R/I$. There is a canonical homomorphism of graded $S$-algebras:

$$
\lambda^S_\ast : \wedge^S H_1(E) \rightarrow H_\ast(E),
\text{cls}(z_1) \wedge \cdots \wedge \text{cls}(z_m) \mapsto \text{cls}(z_1 \wedge \cdots \wedge z_m).
$$

The focus of this paper is on quasi-complete intersection ideals.

Definition. The ideal $I$ is said to be a quasi-complete intersection if $H_1(E)$ is free as an $S$-module
and $\lambda^S_\ast$ is an isomorphism.

As Avramov, Henriques, and Szega [5] note, these ideals were first introduced in Rodicio’s paper
[18] and in his joint work with Blanco and Majadas [9] as ideals having free exterior Koszul homology.
The quasi-complete intersection nomenclature is due to Avramov et al. [5, 1.1].

Like complete intersection ideals, quasi-complete intersections can be described from an ideal-theoretic
perspective: An ideal generated by a sequence of exact elements is necessarily a quasi-complete intersection; see [5, Theorem 3.7] and [15, Theorem 1.8]. The converse does not hold: In [16, Example 4.1], Kustin, Szega, and Vraciu give an example of a quasi-complete intersection which cannot be generated by a sequence of exact elements.

We study homological characterizations of quasi-complete intersections. Our primary tool in this study is the Tate construction. This complex is the second step in a Tate resolution of $S$ over $R$, i.e., it is the result of adjoining (to the Koszul complex $E$) variables of degree two to annihilate the degree one homology of $E$; see [20, §2]. In Section 1, we recall the properties of the Tate construction and the related Cartan construction.

Blanco, Majadas and Rodicio [10, Theorem 1] provide a characterization of quasi-complete intersection ideals as follows. Let $z$ denote a set set of cycles whose homology classes generate $H_1(E)$ and let $T$ be the Tate construction on $f$ and $z$. Then $H_i(T) = 0$ for all $i > 0$ if and only if $I$ is

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a quasi-complete intersection and \( z \) represents a basis for the \( S \)-module \( H_1(E) \). In Section 2 we strengthen one direction of this characterization with the following result (Theorem 2.1):

**Theorem A.** Suppose \( f = \{f_1, \ldots, f_n\} \) and set \( b = n - \text{depth}(I, R) \). If \( H_i(T) = 0 \) for \( i = 2, \ldots, b + 2 \), then \( I \) is a quasi-complete intersection and \( z \) represents a basis for the \( S \)-module \( H_1(E) \).

In Section 3 we provide another characterization of quasi-complete intersection ideals (Theorem 3.1). We detect the quasi-complete intersection property from a band of vanishing of width \( b + 1 \) (as in Theorem A) and with an additional assumption on the size of \( z \) we have flexibility in the location of the band.

In Section 4 we utilize the fact (established by Assmus [1, Theorem 2.7]) that the maximal ideal of a local ring is a quasi-complete intersection if and only if the ring is a complete intersection. We obtain the following characterization of (local) complete intersection rings (Theorem 4.4):

**Theorem B.** Let \( x \) be a minimal generating set of \( m \), and set \( b = \text{embdim} R - \text{depth} R \). Suppose \( z = \{z_1, \ldots, z_b\} \) is a set of cycles whose homology classes form a minimal generating set of \( H_1(E) \).

Let \( T \) be the Tate construction on \( x \) and \( z \). The following conditions are equivalent:

1. \( R \) is a complete intersection.
2. There exists an integer \( q \geq 2 \) such that \( H_i(T) = 0 \) for \( i \in \{q, \ldots, q + b - 1\} \).

In particular, the quasi-complete intersection property of \( m \) (equivalently: the complete intersection property of \( R \)) can be detected from a band of vanishing of \( H_2(T) \) of width \( b \) (compared to width \( b + 1 \) of Theorems A and B).

Assmus [1, Theorem 2.7] also characterizes complete intersections as rings for which \( H_2(T) = 0 \). Utilizing results of Halperin [13, Theorem B], Gulliksen [12, Theorem 3.5.1], and Avramov [3, Theorem 2.3], we expand on this characterization with the following result (Theorems 4.7 and 5.5):

**Theorem C.** Suppose that one of the following conditions holds:

1. \( H_i(T) = 0 \) for \( i = 3 \) or 4.
2. \( H_i(T) = 0 \) for some \( i \geq 5 \) and there is a Golod homomorphism from a complete intersection ring onto \( \hat{R} \).

Then \( R \) is a complete intersection.

1. **The Tate Construction**

Throughout this paper, \( R \) is a commutative (not necessarily Noetherian) ring. We recall the construction of two families of complexes, due respectively to Tate [20] and Cartan. We adopt the notation of [12, 20]. In particular, if \( X \) is a differential graded (DG) \( R \)-algebra and \( v \) is a homogeneous cycle of \( X \), then \( X(V | \partial V = v) \) denotes the extension of \( X \) obtained by adjoining a variable \( V \) to annihilate the cycle \( v \). The type of variable depends on the degree of \( v \): If \( |v| \) is odd then \( V \) is an exterior variable, and if \( |v| \) is even then \( V \) is a divided powers variable; see [4, Construction 6.1].

Let \( I \) denote a proper non-zero ideal of \( R \), and \( S = R/I \). We fix a generating set \( f \) of \( I \). Let \( E \) denote the Koszul complex on \( f \), i.e., \( E = K(f; R) \). We have an identification of \( E \) as an extension of \( R \). Let \( u = \{u_f : f \in f\} \) denote a set of degree one exterior variables. Then

\[
E = R(u | \partial u_f = f).
\]

**Construction 1.1. The Tate construction.** Let \( z \) be a set of cycles of degree one such that the homology classes \( \{\text{cls}(z) : z \in z\} \) generate \( H_1(E) \). Let \( w = \{w_z : z \in z\} \) denote a set of degree two divided powers variables. The Tate construction on \( f \) and \( z \), denoted \( T(f; z) \) is

\[
T(f; z) = R(u, w | \partial u_f = f, \partial w_z = z) = E(w | \partial w_z = z).
\]
Let $T$ denote the Tate construction $T(f; z)$; we have the equality $H_1(T) = 0$ and isomorphisms $H_0(T) \cong H_0(E) \cong S$.

The Tate construction $T$ has the following explicit presentation. Let $W$ be a graded $R$-module on the basis $w$, and let $\Gamma^R_p W$ denote the divided powers algebra on $W$. For integers $j(w) \geq 0$ with $p = \sum_{w \in w} j(w)$, the distinct expressions $\prod_{w \in w} w^{j(w)}$ form a basis of $\Gamma^R_p W$. This yields the following presentation of the complex:

\[
T_n = \bigoplus_{2p+q=n} E_q \otimes_R \Gamma^R_p W,
\]

\[
\partial_n^T \left( e \otimes \prod_{w \in w} w^{j(w)} \right) = \partial^E(e) \otimes \prod_{w \in w} w^{j(w)}
\]

\[+(-1)^{|e|} \sum_{w \in w} \left( z' e \otimes w^{j(w) - 1} \prod_{w \neq w'} w^{j(w)} \right) \].

**Remark 1.2.** For a local ring $(R, \mathfrak{m})$, we have a uniqueness property of the Tate construction. Let $f$ denote a minimal generating set of $I$, and let $E$ denote the Koszul complex on $f$. If $z$ is set of degree one cycles whose homology classes form a minimal generating set of $H_1(E)$, then $T(f; z)$ is unique up to isomorphism (see, for example [6, 1.2]). As such, we may (in this context) simply refer to the Tate construction on $I$ without risk of confusion.

**Remark 1.3.** The explicit presentation of the Tate construction $T = T(f; z)$ yields a convergent first-quadrant spectral sequence:

\[
\{d^{pq}_{r} : E^{r}_{pq} \to E^{r}_{p-r,q+r-1} \}_{r \geq 0}; \quad E^{r}_{pq} \Rightarrow H_{p+q}(T).
\]

The $E^0$ and $E^1$ pages are as follows:

\[
E^0_{pq} = E_{q-p} \otimes_R \Gamma^R_p W, \quad E^1_{pq} = H_{q-p}(E) \otimes_S \Gamma^S_p (S \otimes_R W).
\]

Let $H_i(E)$ and let $\Gamma_j$ denote $\Gamma_j^S(S \otimes_R W)$. For $q \geq 0$, the row $E^1_{q\cdot}$ of the $E^1$ page of the spectral sequence is as follows:

\[
0 \leftarrow H_q \leftarrow H_{q-1} \otimes_S \Gamma_1 \leftarrow \cdots \leftarrow H_2 \otimes_S \Gamma_{q-2} \leftarrow H_1 \otimes_S \Gamma_{q-1} \leftarrow \Gamma_q \leftarrow 0
\]

This spectral sequence will be utilized in Section 2; the realization of the Tate construction as an extension of $R$ will appear in Sections 3 through 5.

We now recall the prototype for the Tate construction: the Cartan construction.

**Construction 1.4.** The Cartan construction. Let $B$ denote a DG $R$-algebra with differential $\partial^B = 0$. Let $y$ denote a set of generators of $B_1$, and let $w = \{w_y : y \in y\}$ denote a set of degree two divided powers variables. The Cartan construction $C$ on $B$ is the extension $C = B \langle w | \partial w_y = y \rangle$.

**Remark 1.5.** Let $C$ be the Cartan construction on $B$. Then $C$ is bigraded:

\[
C_{p,q} = B_{q-p} \otimes_R \Gamma^R_p W, \quad C_n = \bigoplus_{p+q=n} C_{p,q}.
\]

Moreover $C$ decomposes into strands $C_{s,q}$:

\[
0 \leftarrow B_q \leftarrow B_{q-1} \otimes_R \Gamma^R_1 W \leftarrow \cdots \leftarrow B_1 \otimes_R \Gamma^R_{q-1} W \leftarrow \Gamma^R_q W \leftarrow 0.
\]
As such, we have a decomposition of the homology of \( C \):

\[
H_n(C) = \bigoplus_{p+q=n} H_p(C_{*,q}).
\]

**Remark 1.7.** Let \( G \) be a free \( R \)-module on a basis \( g \). Set \( B = \wedge^* G \), and consider \( B \) as a DG algebra with differential \( \partial^B = 0 \); note that \( H(B) = B \). Let \( C \) be the Cartan construction on \( B \). Then \( H_p(C_{*,q}) = 0 \) for all \( (p,q) \neq (0,0) \).

Indeed, \( g \) is regular on \( B \) (in the sense of [4, §6]) so that [4, Proposition 6.1.7] yields an isomorphism

\[
\frac{B}{(g)B} \cong H(C).
\]

In light of this and the equalities \((g)B = B_{\geq 1} \) and \( B_0 = R \) we have \( H_n(C) = 0 \) for \( n > 0 \) and \( H_0(C) = R \). Now (1.6) yields desired result.

### 2. Low-degree vanishing of \( H_*(T) \)

Recall that \( E \) is the Koszul complex on a fixed generating set \( f \) of \( I \). Throughout this section, \( z \) denotes a set of degree one cycles such that the homology classes \( \{\text{cls}(z) : z \in z\} \) generate \( H_1(E) \). Let \( T \) denote the Tate construction on \( f \) and \( z \) given by Construction 1.1. In this section, we prove the following result (Theorem 3.1):

**Theorem 2.1.** Suppose that \( I = (f) \) is a proper, non-zero ideal of \( R \), and set \( b = \max\{i : H_i(E) \neq 0\} \). Suppose \( z \) is a set of cycles whose homology classes generate \( H_1(E) \). Let \( T \) be the Tate construction on \( f \) and \( z \). The following conditions are equivalent:

1. \( I \) is a quasi-complete intersection ideal and \( z \) represents a basis of the \( S \)-module \( H_1(E) \).
2. \( H_i(T) = 0 \) for all \( i > 0 \).
3. \( H_i(T) = 0 \) for \( i = 2, \ldots, b + 2 \).

**Remark 2.2.** When \( R \) is Noetherian, the integer \( b \) can be computed as follows: For \( I = (f_1, \ldots, f_c) \) and \( I \neq I^2 \), [17, Theorem 16.8] yields that \( b = c - \text{depth}(I, R) \), where \( \text{depth}(I, R) \) denotes the length of a maximal \( R \)-sequence contained in \( I \).

We begin by noting a relationship between the properties of the map \( \lambda^S_i \) and the homology of \( T \). The map \( d_{1}^{1:1} \) of the spectral sequence of Remark 1.3 is given by

\[
d_{1}^{1:1} : S \otimes_R W \to H_1(E), \quad s \otimes w_i \mapsto s \text{cls}(z_i).
\]

The construction of \( T \) yields that \( d_{1}^{1:1} \) is surjective, \( S \otimes_R W \) free over \( S \), and \( H_1(T) = 0 \). In addition, \( \lambda^S_i : \wedge^S_1 H_1(E) \to H_1(E) \) is the identity map.

**Proposition 2.3.** Let \( k \geq 1 \) be an integer. The following statements are equivalent:

1. \( H_i(T) = 0 \) for \( i = 2, \ldots, k + 1 \)
2. \( d_{1}^{1:1} \) is an isomorphism, \( \lambda^S_i \) is an isomorphism for \( i = 2, \ldots, k \), and \( \lambda^S_{k+1} \) is surjective.

**Proof.** (1) \( \implies \) (2): We first establish that \( d_{1}^{1:1} : S \otimes_R W \to H_1(E) \) is injective (and is thus an isomorphism of \( S \)-modules). Recall that the terms \( E_{p,q}^0 \) are non-zero only for \( (p,q) \) satisfying \( q \geq p \geq 0 \). Thus \( E_{0,1}^2 = 0 \), and so \( E_{1,1}^2 = 0 \). Moreover, \( E_{1,1}^2 = E_{1,1}^\infty \) is (isomorphic to) a subquotient of \( H_2(T) \), so that \( d_{1}^{1:1} \) is injective, as desired.

We now focus on the properties of the maps \( \lambda_i \) described in (2). In addition, we show that the following condition holds:

(3) \( E_{0,k+1}^2 = 0 \) and \( E_{p,q}^2 = 0 \) for all \( (p,q) \neq (0,0) \) with \( 0 \leq p \leq q \leq p + k - 1 \).
We will establish (2) and (3) by induction on \( k \).

Suppose \( k = 1 \). We begin with (3) and show that \( E^2_{q,0} = 0 \) for all \( q \geq 1 \). Let \( C \) be the Cartan construction on the free \( S \)-module \( H_1(E) \) and let \( D^q \) denote the strand \( C_{s,n} \) (see Construction 1.4). Let \( \Gamma \) denote \( \Gamma^S_1(W \otimes R) \). For each \( q \geq 1 \), we have morphisms relating the row \( E^1_{q} \) of the \( E^1 \) page of the spectral sequence to the strand \( D^q \):

\[
E^1_{q} : \quad \cdots \xrightarrow{} H_2(E) \otimes_S \Gamma_{q-2} \xleftarrow{} H_1(E) \otimes_S \Gamma_{q-1} \xleftarrow{} H_1(E) \otimes_S \Gamma_{q} \xleftarrow{} H_1(E) \otimes_S \Gamma_{q-1} \xrightarrow{} 0
\]

\[
D^q : \quad \cdots \xleftarrow{} \wedge^S_2 H_1(E) \otimes_S \Gamma_{q-2} \xleftarrow{} H_1(E) \otimes_S \Gamma_{q-1} \xleftarrow{} H_1(E) \otimes_S \Gamma_{q} \xrightarrow{} H_1(E) \otimes_S \Gamma_{q-1} \xrightarrow{} 0
\]

From this diagram we conclude that \( E^2_{q,0} = H_q(D^q) \). But Remark 1.7 yields that \( H_q(D^q) = 0 \) for all \( q \geq 1 \), and so \( E^2_{q,0} = 0 \) for all \( q \geq 1 \), as desired. It remains to show that \( E^2_{0,2} = 0 \). We have \( E^2_{0,2} = E^2_{0,2} \); this is (isomorphic to) a subquotient of \( H_2(T) \), so that \( E^2_{0,2} = 0 \).

For (2) note that \( \lambda^S_1 \) is the identity map on \( H_1(E) \); we now show that \( \lambda^S_2 \) is surjective. Note that \( E^1_{1,2} = 0 \), and so \( E^2_{o,2} = \text{Coker } d^1_{1,2} \). But \( E^2_{0,2} = 0 \), and so \( d^1_{1,2} \) is surjective. We have the following commutative diagram with exact rows:

\[
E^1_{1,2} : \quad 0 \xleftarrow{} H_2(E) \xrightarrow{d^1_{1,2}} H_1(E) \otimes_S \Gamma_1
\]

\[
D^2 : \quad 0 \xleftarrow{} \wedge^S_2 H_1(E) \xrightarrow{} \wedge^S_2 H_1(E) \otimes_S \Gamma_1
\]

Thus \( \lambda^S_2 \) is surjective, as desired.

Suppose now that \( k \geq 2 \). By construction and by induction \( E^2_{1,k} = E^1_{1,k} \); this is (isomorphic to) a subquotient of \( H_{k+1}(T) \), and so \( E^2_{1,k} = 0 \). Similarly, \( E^2_{0,k} = 0 \). These equalities yield the following commutative diagram with exact rows:

\[
E^1_{s,k} : \quad 0 \xleftarrow{} H_k(E) \xleftarrow{} H_{k-1}(E) \otimes_S \Gamma_1 \xleftarrow{} H_{k-2}(E) \otimes_S \Gamma_2
\]

\[
D^k : \quad 0 \xleftarrow{} \wedge^S_k H_1(E) \xrightarrow{} \wedge^S_k H_1(E) \otimes_S \Gamma_1 \xleftarrow{} \wedge^S_{k-2} H_1(E) \otimes_S \Gamma_{k-2}
\]

An application of the four lemma now gives that \( \lambda^S_k \) is an isomorphism.

To establish (3) it remains to show that \( E^2_{pq} = 0 \) for all \( (p,q) = (q - k + 1, q) \), where \( q \geq k + 1 \). For each such \( q \), we have the following commutative diagram:

\[
E^1_{q} : \quad H_k(E) \otimes_S \Gamma_{q-k} \xleftarrow{} H_{k-1}(E) \otimes_S \Gamma_{q-k+1} \xleftarrow{} H_{k-2}(E) \otimes_S \Gamma_{q-k+2}
\]

\[
D^q : \quad \wedge^S_k H_1(E) \otimes_S \Gamma_{q-k} \xleftarrow{} \wedge^S_{k-1} H_1(E) \otimes_S \Gamma_{q-k+1} \xleftarrow{} \wedge^S_{k-2} H_1(E) \otimes_S \Gamma_{q-k+2}
\]

Hence we have an isomorphism \( E^2_{q-k+1,q} \cong H_{q-k+1}(D^q) \). Noting that \( q - k + 1 \geq 2 \), Remark 1.7 yields \( H_{q-k+1}(D^q) = 0 \), and so \( E^2_{q-k+1,q} = 0 \) for each \( q \geq k + 1 \).
For (2), it remains to show that \( \lambda^S_{k+1} \) is surjective. As \( E^1_{1,k+1} = 0 \), we have \( \mathrm{Coker} \, d^1_{1,k+1} = E^2_{0,k+1} \). But \( E^2_{0,k+1} = E^\infty_{0,k+1} \); this is (isomorphic to) a subquotient of \( H_{k+1}(T) \), so that \( E^2_{0,k+1} = 0 \). This in turn yields that \( d^1_{1,k+1} \) is surjective. From this, we have the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
E^1_{*k+1} & : & 0 \leftarrow H_{k+1}(E) \leftarrow H_k(E) \otimes_S \Gamma_1 \\
 & \uparrow \lambda^S_{k+1} & \cong \uparrow \lambda^S_k \otimes \Gamma_1 \\
D^k_{*+1} & : & 0 \leftarrow \wedge^S_{k+1} H_1(E) \leftarrow \wedge^S_k H_1(E) \otimes_S \Gamma_1 \\
\end{array}
\]

(2.7)

From this, we conclude that \( \lambda^S_{k+1} \) is surjective.

(2) \( \implies \) (1): As above, let \( C \) denote the Cartan construction on the free \( S \)-module \( H_1(E) \). First, we have that \( E^2_{p,q} = 0 \) for \( (p,q) = (0,k) \) and for all \( (p,q) \neq (0,0) \) with \( 0 \leq p \leq q \leq p + k - 1 \). Indeed, by utilizing commutative diagrams analogous to (2.4), (2.5), and (2.6), we have that \( E^2_{p,q} \) is isomorphic to \( H_p(D^q) \) for \( (p,q) = (0,k) \) and for all \( (p,q) \neq (0,0) \) with \( 0 \leq p \leq q \leq p + k - 1 \). Consequently, Remark 1.7 yields that \( E^2_{p,q} = 0 \) for all such \( (p,q) \). Second, noting that \( \lambda^S_{k+1} \) is surjective, we see from a diagram analogous to (2.7) that \( E^2_{0,k+1} = 0 \) as well.

In particular, this vanishing of \( E^2 \) in this region yields that \( E^\infty_{p,q} = E^2_{p,q} = 0 \) for all \( (p,q) \) satisfying \( 0 < p + q \leq k + 1 \). For each such \( (p,q) \) we have a finite filtration

\[
(2.8) \quad 0 = F_{-1} H_{p+q} \subseteq F_0 H_{p+q} \subseteq \cdots \subseteq F_{p+q} H_{p+q} = H_{p+q}(T).
\]

Each quotient of consecutive terms has the form

\[
\frac{F_p H_{p+q}}{F_{p-1} H_{p+q}} \cong E^\infty_{p,q} = 0.
\]

We therefore conclude that each containment in (2.8) is an equality, and hence \( H_{p+q}(T) = 0 \) for all \( 0 < p + q \leq k + 1 \), so that \( H_i(T) = 0 \) for all \( i = 2, \ldots, k + 1 \). \( \square \)

With Proposition 2.3 in hand, we are now prepared to prove Theorem 2.1. Note that the result of Blanco, Majadas, and Rodicio ([10, Theorem 1]) establishes the equivalence (1) \( \iff \) (2).

Proof of Theorem 2.1: (1) \( \implies \) (2) was established by Tate [20], and (2) \( \implies \) (3) is clear.

(3) \( \implies \) (1): By Proposition 2.3, \( H_1(E) \) is free as an \( S \)-module via \( d^1_{1,1} : S \otimes_R W \to H_1(E) \), so that \( H_1(E) \) has the desired basis. Moreover, \( \lambda^S_i \) is an isomorphism for \( i = 1, 2, \ldots, b + 1 \). As \( H_{b+1}(E) = 0 \), we have that \( \wedge^S_{b+1} H_1(E) = 0 \), and so \( \mathrm{rank}_S H_1(E) \leq b \). Then for each \( i > b + 1 \) we have the equality \( \wedge^S H_1(E) = 0 \) and \( \lambda^S_i \) is an isomorphism (of zero modules). \( \square \)

A result of Kustin, Šega, and Vraciu ([16, Lemma 1.7]) provides (in the local case) an analogous classification for two-generated quasi-complete intersection ideals in terms of the vanishing of the homology of the Tate construction and a double annihilator condition.

The following construction provides the framework for the double annihilator condition.

Construction 2.9 ([16, 1.4]). Fix a basis \( v_1, \ldots, v_n \) of \( E_1 \) with \( \partial^E(v_i) = f_i \). Suppose that \( z = \{z_1, \ldots, z_n\} \) is a set of degree one cycles of \( E \) such that the homology classes \( \{\mathrm{cls}(z_i)\} \) minimally generate \( H_1(E) \). Then there exist \( \{a_{ij} : i, j \in [1, n]\} \subset R \) such that

\[
z_i = \sum_{j=1}^{n} a_{ij} v_j.
\]
Let $A$ denote the matrix $(a_{ij})$ and set $\Delta = \det A$. Then the map $\lambda^n_\mathcal{S} : \wedge^n_\mathcal{S} H_1(E) \to H_n(E)$ is given by

$$\text{cls}(z_1) \wedge \cdots \wedge \text{cls}(z_n) \mapsto \Delta v_1 \cdots v_n.$$ 

The equality $\nu_R(H_1(E)) = n$ holds whenever $I$ is a quasi-complete intersection with $\text{depth}(I, R) = 0$; see [5, 1.2].

**Lemma 2.10 ([16, Lemma 1.7]).** Suppose $\nu_R(I) = 2$ and $\text{depth}(I, R) = 0$. Then the following statements are equivalent:

1. $I$ is a quasi-complete intersection.
2. $H_2(T) = 0$, $(0 : R I) = (\Delta)$, and $(0 : R \Delta) = I$, where $\Delta$ is as defined in Construction [2.9].

**Remark 2.11.** Suppose $I = (f)$ is principal and there exists $g \in R$ with $(0 : R f) = (g)$. If $H_2(T(f; g)) = 0$, then $I$ is a quasi-complete intersection. Indeed, the Tate construction $T(f; g)$ has the following form:

$$0 \leftarrow f \leftarrow R \leftarrow g \leftarrow R \leftarrow f \leftarrow \cdots$$

The hypothesis that $H_2(T(f; g)) = 0$ yields that $(0 : R f) = (g)$, and consequently $H_i(T(f; g)) = 0$ for all $i > 0$ and $I$ is a quasi-complete intersection.\footnote{In this context, the pair $(f, g)$ is an exact pair in the sense of Kiełpiński, Simson, and Tyc [15, Definition 1.1].}

The following result illustrates another case in which the vanishing of $H_2(T)$ is sufficient to detect the quasi-complete intersection property of $I$.

**Proposition 2.12.** Suppose $I = (f)$ and $\bigcap_{i \geq 1} (f^i) = (0)$. If $H_2(T) = 0$, then $I$ is a quasi-complete intersection.

**Proof.** Let $E$ denote the Koszul complex $K(f; R)$. Suppose $z$ is a set of non-zero elements of $R$ such that $(z) = \text{ann}_R(f) = H_1(E)$. Let $T$ denote the Tate construction $T(f; z) = E[[w | \partial w_z = z]]$. In this context the differentials $\partial_2^T : W \to R$ and $\partial_3^T : W \to W$ are defined on basis elements by $\partial_2^T(w_z) = z$ and $\partial_3^T(w_z) = fw_z$.

It will suffice to show that $z = \{z\}$. Suppose not, and pick distinct generators $z, z' \in z$. We will show by induction that $z, z' \in (f^i)$ for all $i \geq 1$.

Note that $Z_2(T)$ contains the non-zero cycle $\zeta = z'w_z - zw_{z'}$. As $H_2(T) = 0$ we have that $\zeta$ is a boundary. Thus $z' = r'f$ and $z = rf$ for $r, r' \in R$, so that $z, z' \in (f)$. Suppose now that $z, z' \in (f^i)$ for some $i \geq 1$. Then there exist $s, s' \in R$ with $z = sf^i$ and $z = s'f^i$. Then $Z_2(T)$ contains the non-zero cycle $sw_z - s'w_{z'}$, so that there exists $t, t' \in R$ with $s = tf$ and $s' = tf'$. Thus $z = tff^i$ and $z' = t'ff^i$, so that $z, z' \in (f^{i+1})$, completing the induction.

Therefore $z = 0 = z'$, a contradiction. \qed

3. Vanishing of homology of DG algebras

In Theorem [2.1], we see a situation in which a band of vanishing of $H_i(T)$ implies that $H_i(T) = 0$ for all $i > 0$. In this section we prove the following result, which continues this theme.

**Theorem 3.1.** Set $b = \max\{i : H_i(E) \neq 0\}$, and suppose $z = \{z_1, \ldots, z_b\}$ is a set of cycles whose homology classes generate $H_1(E)$. Let $T$ be the Tate construction on $f$ and $z$. If there exists an integer $q \geq 2$ with $H_i(T) = 0$ for $i \in \{q, \ldots, q + b\}$, then $I$ is a quasi-complete intersection.

The hypothesis that $H_1(E)$ can be generated by $b$ elements means that the size of $H_1(E)$ is compatible with $I$ being a quasi-complete intersection in the following sense:

**Remark 3.2.** If $I$ is a quasi-complete intersection ideal, then $\text{rank}_S H_1(E) = b$. Indeed, the isomorphism $\lambda^n_\mathcal{S}$ yields the equality $b = \max\{i : \wedge^n_\mathcal{S} H_1(E) \neq 0\}$.\footnote{In this context, the pair $(f, g)$ is an exact pair in the sense of Kiełpiński, Simson, and Tyc [15, Definition 1.1].}
We now develop conditions, expressed as a band of vanishing of homology, under which an extension formed by the adjunction of variables of degree two exhibits eventually periodic or eventually vanishing homology. We adopt the notation of [4] §6. For integers $i \leq j$, let $[i, j]$ denote \{i, i+1, \ldots, j\}.

**Lemma 3.3.** Let $A$ denote a DG $R$-algebra. Suppose that $\{z_1, \ldots, z_m\}$ is a set degree one cycles of $A$. Put $A_0 = A$ and for $1 \leq j \leq m$ put $A_j = A_{j-1}[w_j \mid \partial w_j = z_j]$. Let $q$ and $b$ be integers.

(1) Suppose $H_i(A_m) = 0$ for all $i \in [q, q + m]$. Then for each $j$ we have $H_i(A_j) = 0$ for all $i \in [q + m - j, q + m]$.

Suppose further that $H_i(A) = 0$ for all $i > b$.

(2) $H_i(A_1) \cong H_{i+2}(A_1)$ for all $i \geq b$, i.e., $H_*(A_1)$ is periodic of period 2 beginning in degree $b$.

(3) If $q \geq b+1-m$ and $H_i(A_m) = 0$ for all $i \in [q, q+m]$, then $H_i(A_m) = 0$ for all $i \geq b+1-m$.

**Proof.** For (1), by induction we may assume $m = 1$. Then $H_i(A_1) = 0$ for $i \in \{q, q+1\}$. The equality $H_{q+1}(A) = 0$ now follows from immediately from the following portion of long exact sequence in homology associated with Tate’s exact homology triangle [4, Remark 6.1.6]:

$$\cdots \to H_{q+1}(A_1) \to H_{q+2}(A) \to H_{q+2}(A_1) \to H_q(A_1) \to H_{q+1}(A) \to H_{q+1}(A_1) \to \cdots$$

(3.4)

The result of (2) also follows from [4] Remark 6.1.6: For each $i \geq b$ we have an isomorphism $H_{i+2}(A_1) \cong H_i(A_1)$, so that $H_*(A_1)$ is eventually periodic of period 2. The extremal case occurs when $i = b$, namely $H_{b+2}(A_1) \cong H_b(A_1)$, so that the periodicity begins in the desired position.

For (3), we proceed by induction on $m$. Consider the case $m = 1$. By (2), the vanishing of $H_i(A)$ for $i > b$ yields that $H_*(A_1)$ is periodic of period 2 beginning in degree $b$. By hypothesis, $H_i(A_1) = 0$ for $i \in \{q, q+1\}$. As $q \geq b$, we have that one representative from each of the two isomorphism classes of $H_{\geq b}(A_1)$ vanishes, so that $H_i(A_1) = 0$ for all $i \geq b$.

Suppose now that for each $1 \leq a < m$ the statement holds for the adjunction of $a$ variables of degree two, and that $H_i(A_m) = 0$ for all $i \in [q, q+m]$. By (1), $H_i(A_{m-1}) = 0$ for all $i \in [q+1, q+m]$. By induction, we have that $H_i(A_{m-1}) = 0$ for all $i \geq b+1-(m-1)$, so that (2) yields that $H_*(A_m)$ is periodic of period 2 starting in degree $b+1-m$. As $q \geq b+1-m$, we have that (at least) one representative from each of the two isomorphism classes of $H_{\geq b+1-m}(A_m)$ vanishes, which completes the proof. □

**Remark 3.5.** The vanishing guaranteed by Lemma 3.3 begins at a position independent of the location of the band of vanishing. In particular, this yields the following: If $H_i(B) = 0$ for all $i \gg 0$, then $H_i(B) = 0$ for all $i \geq b + 1 - m$.

**Remark 3.6.** In the more general case where the extension $B$ is formed by the adjunction of variables in a single arbitrary even degree or differing even degrees, one can obtain a description of a region of vanishing which implies the eventual vanishing of the homology of such an extension.

**Proof of Theorem 3.1.** Note that $T = E\langle w_1, \ldots, w_b \mid \partial w_i = z_i \rangle$, where $|z_i| = 1$. Lemma 3.3(3) now yields that $H_i(T)$ for all $i \geq 1$. Thus $T$ is acyclic, and $I$ is a quasi-complete intersection. □

4. **Characterizing complete intersections**

In this section, $(R, \mathfrak{m}, k)$ is a (Noetherian) local ring.
Definition 4.1. We say that $R$ is a complete intersection if its $\mathfrak{m}$-adic completion $\widehat{R}$ can be written as a quotient of a (complete) regular local ring by a regular sequence.

A result of Assmus [1] Theorem 2.7 yields that $R$ is a complete intersection if and only if $\mathfrak{m}$ is a quasi-complete intersection ideal. Assmus’ result does not use the quasi-complete intersection terminology: The condition is stated as “$H(E)$ is the exterior algebra on $H_1(E)$”. As Avramov, Henriques, and Søegaard [5, §1] note, the existence of some isomorphism of graded $S$-algebras

$$\lambda : H(E) \xrightarrow{\cong} \wedge^S_h H_1(E)$$

guarantees the quasi-complete intersection property.

Let $T$ denote Tate construction on $\mathfrak{m}$; see Remark 4.2. In this section, we study complete intersection rings by applying the results of Sections 2 and 3. We show that, compared to the non-maximal case, the quasi-complete intersection property of $\mathfrak{m}$ (and hence the complete intersection property of $R$) can be detected from a smaller band of vanishing of $H_*$. Hereafter, the size of a minimal generating set of an $R$-module $M$ is denoted $\nu_R(M)$.

We begin by outlining a construction which will allow us to relate a Tate construction over local ring to a Tate construction over a quotient.

Construction 4.2. The Tate construction on $R$. Assume that $R$ is complete. There exists a regular local ring $(Q, n, k)$ and an ideal $J \subset n^2$ such that $R = Q/J$. Furthermore, $b = \nu_Q(J)$; see, for example, [1] pp 196-197. Select a maximal $Q$-sequence $a_1, \ldots, a_h$ in $J$ so that the images $\{\overline{a_i}\}$ in $J/nJ$ are linearly independent over $Q/n$; we may extend the sequence to a minimal generating set $a_1, \ldots, a_b$ of $J$. Put $J' = (a_1, \ldots, a_h)$ and let $(Q', n')$ denote $(Q/J', n/J')$.

Let $K$ denote the Koszul complex on a minimal generating set of $n$ and let $E'$ denote the Koszul complex on a minimal generating set of $n'$. As before, $h = \nu_{Q'}(H_1(E'))$. Let $z' = z'_1, \ldots, z'_h$ denote the set of cycles given by the construction in [1] pp 196-197; their homology classes form a minimal generating set for $H_1(E')$. Moreover, letting $z_1, \ldots, z_h$ denote their images in $E = E' \otimes_{Q'} R$, the same construction yields that these images extend to a set of cycles $z = z_1, \ldots, z_h$ whose homology classes form a minimal generating set of $H_1(E)$.

Let $F'$ denote the Tate construction on $E'$ and $z'$, so that $F' = E'(w_1, \ldots, w_h | \partial w_i = z'_i)$. Set $F = F' \otimes_{Q'} R = E(w_1, \ldots, w_h | \partial w_i = z_i)$. By construction, $Q'$ is a complete intersection; a result of Assmus (1, Theorem 2.7) yields that $F'$ is a minimal $Q'$-free resolution of $k$, and thus $\text{Tor}^Q_i(R, k) = H_i(F)$. Let $T$ denote that Tate construction on $E$ and $z$. Then $T = F(w_{h+1}, \ldots, w_b | \partial w_i = z_i)$.

Let $\pi$ denote the natural surjection $Q' \to R$; we note a connection between $\text{Ker} \pi$ and $\text{pd}_{Q'} R$.

Remark 4.3. By Construction 4.2 $\text{Ker} \pi$ contains only zerodivisors. A result of Auslander and Buchsbaum [2, Proposition 6.2] now yields the implication $\text{pd}_{Q'} R < \infty \implies (0 : Q' R) = 0$.

The following result (Theorem 3.1) is the improvement of Theorem 3.1.

Theorem 4.4. Suppose there exists an integer $q \geq 2$ such that $H_i(T) = 0$ for $i = [q, q + b - 1]$. Then $R$ is a complete intersection.

Proof. Here we follow the strategy of Gulliksen [11]. Without loss of generality, we may assume that $R$ is complete. Recall the notation of Construction 4.2. Let $\pi : Q' \to R$ be the natural surjection. We will show that $\text{Ker} \pi = 0$; by Remark 4.3 it will be enough to show that $\text{pd}_{Q'} R < \infty$.

Recall that $\text{Tor}^Q_i(R, k) = H_i(F)$. By hypothesis, there exists an integer $q \geq 2$ such that $H_i(T) = 0$ for $i = [q, q + b - 1]$. Noting that we have obtained $T$ from $F$ by adjoining at most $b - 1$ variables of degree two, Lemma 3.3 (1) yields that $H_{q+b-1}(F) = 0$. This implies that $\text{Tor}^Q_{q+b-1}(R, k) = 0$ for some $q \geq 2$. Hence, $\text{pd}_{Q'} R < \infty$, completing the proof. \qed
Let $R(X)$ denote an acyclic closure of $k$ over $R$ and order the variables $X$ such that $|x_i| \leq |x_j|$ for $i < j$; see [4] Construction 6.3.1. Fix an integer $p$ and let $Y$ denote the extension of $R(x_i : i \leq p)$.

We note that the following result appears implicitly in work of Gulliksen [11]

**Proposition 4.5.** Let $F$ be as in Construction 4.2 and suppose that $F \subseteq Y$. If $H_i(Y) = 0$ for all $i \gg 0$, then $R$ is a complete intersection.

**Proof.** The DG-algebra $Y$ satisfies the conditions of [11] Lemma 1. Now $H_i(Y) = 0$ for all $i \gg 0$ and $Y$ is obtained from $F$ by an adjunction of (finitely many) variables, so a repeated application of [11] Lemma 2 yields $H_i(F) = 0$ for all $i \gg 0$. But $H_i(F) = \text{Tor}_i^Q(R,k)$, so that $\text{pd}_Q R < \infty$. Consequently, Remark 4.3 yields that $R$ is a complete intersection.

In particular, the eventual vanishing of $H_\ast(T)$ is equivalent to the complete intersection property of $R$ (i.e., the quasi-complete intersection property of $m$).

Assmus [1] Theorem 2.7] establishes that the complete intersection property of $R$ is equivalent to the vanishing of $H_2(T)$. We now develop the tools needed to extend on this result to show that the vanishing of $H_3(T)$ or $H_4(T)$ also detects the complete intersection property.

The following lemma highlights two situations in which the adjunction of variables to annihilate a non-zero homology class preserves the vanishing of homology in a higher degree.

**Lemma 4.6.** Let $A$ be a DG $R$-algebra and assume that $H_0(A) = k$. Let $i$ be an integer, and suppose that $H_i(A) \neq 0$. Let $z$ be a cycle representing a non-zero homology class in $H_i(A)$ and set $B = A \langle w \mid \partial w = z \rangle$.

1. If $i \geq 2$ is even and $H_1(A) = 0 = H_{i+2}(A)$, then $H_1(B) = 0 = H_{i+2}(B) = 0$.
2. If $H_i+1(A) = 0$, then $H_{i+1}(B) = 0$.

**Proof.** For (1), the equality $H_1(B) = 0$ is clear, and the equality $H_{i+2}(B) = 0$ follows immediately from a portion of the exact sequence from [4] Remark 6.1.5:

$$
\cdots \rightarrow H_{i+2}(A) \rightarrow H_{i+2}(B) \xrightarrow{H_{i+1}(\partial)} H_1(A) \rightarrow \cdots
$$

Let $\zeta$ denote $\text{cls}(z)$. For (2), suppose first that $i$ is even.

We consider the following portion of exact sequence in homology of [4] Remark 6.1.5:

$$
\cdots \rightarrow H_{i+1}(A) \rightarrow H_{i+1}(B) \xrightarrow{H_i(\partial)} H_0(A) \xrightarrow{\zeta} H_i(A) \rightarrow \cdots
$$

Multiplication by $\zeta$ is injective on $H_0(A)$, so that $H_{i+1}(B) = 0$, as desired.

In the case where $i$ is odd, the relevant portion of the exact sequence in homology from [4] Remark 6.1.6] is the following:

$$
\begin{array}{c}
H_{i+1}(A) \rightarrow H_{i+1}(B) \\
\| \rightarrow H_0(B) \\
\| \rightarrow H_i(A) \\
\| \rightarrow H_i(B) \rightarrow H_{-1}(B)
\end{array}
$$

By construction $H_i(\zeta) = 0$, and so $H_i(\partial)$ is not injective. Thus $\partial_{i+1}$ is not the zero map and so $\partial_{i+1}$ is injective. Thus $H_{i+1}(B) = 0$. □

\(^2\) An example of such an extension is a *partial acyclic closure* $R(X_{\leq n})$. 
In the next result (Theorem C, condition (1)), we make use of the deviations $\varepsilon_n(R)$ of $R$, for which we use [4, §7] as a reference. Let $T$ denote the Tate construction on $E$; see Remark 1.2.

**Theorem 4.7.** If $H_i(T) = 0$ for some $i = 3$ or 4, then $R$ is a complete intersection.

*Proof.* We may assume that $R$ is not a complete intersection, so that $H_2(T) \neq 0$.

If $H_3(T) \neq 0$, then we apply Lemma 1.6 (2) and adjoin variables of degree three to obtain a partial acyclic closure $B$ of $k$ with $H_i(B) = 0$ for $i \in \{1, 2, 3\}$. This yields $\varepsilon_4(R) = 0$, so that by a result of Gulliksen [12, Theorem 3.5.1], $R$ is a complete intersection, a contradiction.

Suppose now that $H_4(T) = 0$. We adjoin variables of degrees 3 and 4; applying Lemma 1.6 (1) and (2), we obtain a partial acyclic closure $V$ of $k$ with $H_i(V) = 0$ for $i = 1, 2, 3, 4$, so that $\varepsilon_5(R) = 0$. Now Halperin [13, Theorem B] gives that $R$ is a complete intersection, a contradiction. \qed

5. Rigidity of the Tate construction

In this section $(R, \mathfrak{m}, k)$ is a local ring and let $T$ denote the Tate construction on $\mathfrak{m}$.

Previous work ([4, Theorem 2.7]) and the work of this paper (Theorem 4.7) suggest the following question:

**Question 5.1.** Does the implication

$$H_i(T) = 0 \text{ for some } i \geq 0 \implies R \text{ is a complete intersection}$$

hold for every local ring $R$?

Suppose that $\varphi : Q \to R$ is a surjective homomorphism of local rings and $M$ is a finite $R$-module. Recall the Poincaré series of $M$ over $R$:

$$P^R_M(t) = \sum_{n=0}^{\infty} \beta^R_n(t)t^n \in \mathbb{Z}[[t]].$$

The following result relates the Betti numbers of $M$ over $R$ and $Q$.

**Proposition 5.2.** [4, Proposition 3.3.2] Then there is a coefficientwise inequality of formal power series

$$(5.3) \quad P^R_M(t) \preceq \frac{P^Q_M(t)}{1 - t(P^Q_R(t) - 1)}.$$  

We present a class of rings for which Question 5.1 holds. This class is defined in terms of Golod homomorphisms, for which we use [3,4] as references.

**Definition 5.4.** [4, §3.3] A surjective homomorphism $\varphi : Q \to R$ is called a Golod homomorphism if equality holds in (5.3) for $M = k$.

**Theorem 5.5.** Suppose that there exists a complete intersection ring $Q$ and a Golod homomorphism $\varphi : Q \to \hat{R}$. If $H_i(T) = 0$ for some $i \geq 5$, then $R$ is a complete intersection.

This is Theorem C condition (2).

*Proof.* By [8, Proposition 5.13] we may assume that $\text{depth}_Q(R) = 0$. We endeavor to show that $\text{Ker } \varphi = 0$. By Remark 1.3 it is enough to show that $\text{pd}_Q \hat{R} < \infty$.

Let $F'$ denote the Tate construction on $n$, and put $F = R \otimes_Q F'$. As $Q$ is a complete intersection, we have that $F'$ is a minimal $Q$-free resolution of $k$. Let $A$ denote the trivial extension $k \times H_{\geq 1}(F)$. Then [3, Theorem 2.3] yields that $F$ and $A$ are equivalent as DG-algebras.

Let $y$ be a set of cycles of degree one whose homology classes form a minimal generating set of $H_1(F)$, and let $X$ denote the Tate complex on $A$ and $y$. Then [12, Proposition 1.3.5] yields the equivalence $T \simeq X$. Thus, there exists an integer $i \geq 5$ with $H_i(X) = 0$.

3The indexing convention of the $\varepsilon_n$ differs from that of Gulliksen and Levin [12]: $\varepsilon_3$ of [12] stands for $\varepsilon_4$ of [4].
As the differential on $A$ is trivial, we observe that $X$ exhibits a direct sum decomposition (cf. Remark 1.5):

$$X = \bigoplus_{j \geq 0} D^j,$$

where $D^j$ is the complex

$$0 \leftarrow H_j(F) \xleftarrow{\partial^D_j} H_{j-1}(F) \otimes \Gamma^1 \xleftarrow{\partial^D_j} \cdots \xleftarrow{\partial^D_j} H_1(F) \otimes \Gamma^k \xleftarrow{\partial^D_j} \Gamma^k \leftarrow 0$$

Consequently, we have a decomposition of the homology of $X$:

$$H_k(X) = \bigoplus_{i \geq 0} H_i(D^{k-i}) = \bigoplus_{i=0}^k H_i(D^{k-i}).$$

The equivalence $F \simeq A$ yields that $[H_{k-1}(F)]^2 = 0$, and so the differential $\partial^D_i$ is zero for each $i$ in $\{1, 2, \ldots, j - 1\}$. In light of (5.6), this yields that $H_0(D^k) = H_k(F)$ for each $k \geq 2$, so that $H_k(X)$ contains $H_k(F)$ as a summand for each $k \geq 2$. As such, $H_i(F) = 0$, and so $\text{Tor}_i^Q(R, k) = 0$. Therefore, $\text{pd}_Q R < \infty$, and hence $R$ is a complete intersection. □

**Remark 5.7.** The hypotheses of Theorem 5.5 are satisfied in the following situations:

1. $R$ is a Golod ring,
2. $R$ is Gorenstein and $\text{embdim} R = 4$; see [14, Theorem B],
3. $\text{codepth} R \leq 3$; see [8, Proposition 6.1],
4. $\mathfrak{m}$ has a Conca generator (i.e., there exists $x \in \mathfrak{m}$ such that $x^2 = 0$ and $\mathfrak{m}^2 = x\mathfrak{m}$); see [7, Theorem 1.4],
5. $R$ is a compressed Gorenstein ring of socle degree $s$ and embedding dimension $e$ for $2 \leq s \neq 3$ and $e > 1$; see [19, Theorem 5.1].

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